The pricing of economic risks under time-separable and recursive preferences

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Abstract

Consider an agent with time-separable constant absolute risk aversion preferences who faces an exogenous income process and has access to a riskless saving technology with a constant interest rate. That agent's indirect utility over income streams is a form of Epstein–Zin (1989) preferences. Epstein–Zin preferences have been widely studied in the recent literature because of their implications for the pricing of shocks to expectations of future economic conditions. Identical predictions are naturally and generally obtained also under purely time-separable preferences. The paper's results also point toward a very broad class of analytically solvable models.

One of the primary goals of asset pricing is to understand what fundamental economic risks determine average asset returns. The aim is to understand what risks investors desire to hedge and how much they pay for that insurance. The consumption-based asset pricing literature studies a variety of preference specifications to try to understand risk premia. Models with an intrinsic preference for an early resolution of uncertainty, the preferences of Epstein and Zin (1989, 1991) and Weil (1989) in particular, have gained broad recent attention due to their predictions for the pricing not only of contemporaneous fluctuations, but also of news about the future path of the economy.[1] They can generate a variance risk premium, a premium for news about variation in disaster risk, a premium for news about future growth rates, and can help explain the prices of options and other derivatives.[2]

This paper shows that those predictions for the pricing of economic risks do not actually require intrinsic preferences over uncertainty. That is, they do not require the use of Epstein–Zin preferences over consumption. In fact, for an agent with time-separable preferences, indirect utility over *income* in general takes the form of Epstein–Zin preferences. All the risks described above – variation in growth rates, volatility, and crash risk – among others, that are priced when investors have Epstein–Zin preferences over consumption, are also priced when investors have time-separable preferences.

[1] See also the related recent work on model uncertainty and ambiguity aversion, e.g. Hansen and Sargent (2001), Ju and Miao (2012), and Drechsler (2013).

[2] Representative examples include Weil (1989); Campbell (1993); Bansal and Yaron (2004); Drechsler and Yaron (2011); and Wachter (2013)

The only difference is in measurement: for Epstein–Zin preferences over consumption, it is volatility risk etc. in *consumption* that is priced, whereas under time-separable preferences, those risks are priced when they are part of the *income* process.

The key result in the paper is an analytic solution for an optimal consumption path that does not seem to have been previously discussed explicitly in the literature. It is remarkable not only for being extremely simple and general, but also for suggesting insights into the connection between generalized recursive preferences and optimization. It has long been known that preferences over income streams do not follow the von Neumann–Morgenstern axioms. Specifically, a preference for early resolution of uncertainty about income naturally arises even when utility over consumption is time-separable since the information can be used to form a better consumption plan. What is notable about the results derived here is that they describe a case with an *exact* link between widely studied forms of separable and recursive preferences.

Consider an agent with time-separable constant absolute risk aversion (CARA) preferences over consumption, C_t, of the form $U(C_t) = -\alpha^{-1}\exp(-\alpha C_t)$, where α is the coefficient of absolute risk aversion. The agent has access to a riskless saving technology with gross interest rate R. The agent has an exogenous income process, Y_t. Wealth follows,

$$W_t = RW_{t-1} + Y_t - C_t \tag{1}$$

It is straightforward to confirm that the optimal consumption rule is

$$C_t = (R-1)W_{t-1} + Z_t - \alpha^{-1}\frac{\log \beta R}{R-1} \tag{2}$$

$$Z_t = \frac{(R-1)}{R}Y_t - \frac{1}{\alpha}R^{-1}\log E_t \exp(-\alpha Z_{t+1}) \tag{3}$$

where β is the agent's subjective time discount factor. Z_t is defined recursively and represents the annuity value of human wealth. This solution holds for any income process as long as the recursion defining Z_t exists (a necessary but not sufficient condition is that the conditional cumulant-generating function of Y_t must exist).

Furthermore, one may show that, holding wealth fixed, preferences over income streams are described by Z_t. The variable Z_t is a form of Epstein–Zin (1989) preferences on Y_t. In other words, *for an agent with CARA utility and a riskless savings technology, the consumption function is known and indirect utility over income is exactly Epstein–Zin.*[3]

Now since income can be converted one-for-one into consumption goods, the pricing kernel can be derived either by differentiating the utility function on consumption or the utility function on

[3] Caballero (1990) derives a very similar result in work on puzzles in consumption, but without noting the connection to generalized recursive preferences. He also does not explore the range of processes for which the consumption function can be solved exactly.

income, Z_t, yielding the pricing kernel, M_{t+1},

$$M_{t+1} = \frac{\exp\left(-\alpha Z_{t+1}\right)}{E_t \exp\left(-\alpha Z_{t+1}\right)} \tag{4}$$

This function may be recognized as similar to the typical results obtained for Epstein–Zin preferences over consumption. News about all moments of the distribution of future income growth (the mean, variance, skewness, kurtosis, etc.) is priced.

A similar result may be obtained under power utility, though it requires an approximation to the budget constraint. In that case, the recursion is defined over returns on the wealth portfolio. We thus obtain a broad generalization of the intertemporal CAPM derived by Campbell (1993) in which news about the future distribution of asset returns is priced, again, without Epstein–Zin preferences (which Campbell (1993) used).

So the claim of this paper is that the qualitative pricing of fundamental *economic* risks is highly similar or even identical under time-separable preferences to the intuition that has been developed about generalized recursive preferences. The difference is simply that for Epstein–Zin preferences the relevant variation in future economic conditions is measured by the consumption path, while under time-separable utility, conditions are measured by variation in income or asset returns.

There is a long tradition of attempting to price asset returns based on consumption, since in standard utility-based models the pricing kernel is a function of consumption growth. It is well known, though, that consumption is extremely difficult to measure. There are problems introduced both due to aggregation over time (consumption happens continuously but is measured at lower frequencies) and aggregation across people (the consumption of investors need not be identical to aggregate consumption). Moreover, at higher frequencies, expenditures on consumption goods, even for non-durables like food, need not be identical to the consumption that drives utility. Wong (2015), for example, shows that stockpiling of non-durables is a major source of saving for many households. And the consumption flow obtained from durable goods is not directly observable at all – at best we can infer consumption by assuming a constant flow of services from durables (and this ignores the further complication of modeling the substitutability of durables and non-durables).

An alternative tradition therefore tries to find expressions for the pricing kernel using data on asset returns – e.g. Merton (1969) and Campbell (1993). The results here represent a broad generalization of the results in Campbell (1993) in two directions. In the case of power utility, they allow for much more general models of returns than the cases in Campbell (1993): there are linear solutions available for any affine process for asset returns (up to existence conditions).[4] So while Campbell's (1993) ICAPM allowed only for time variation in volatility, the analysis here can accommodate variation in higher moments, as would arise due to changes in crash risk, for example. Second, they show that one may derive an intertemporal CAPM in which assets are priced based on covariances either with market returns or with income.

[4] Where "affine" here is in the sense of Duffie, Filipovic, and Schachermayer (2003), which is a far broader class than just ARMA models.

In addition to the pricing of economic risks, generalized recursive preferences are well known to have implications for preferences over the timing of the resolution of uncertainty. When there is a consumption-saving decision to make, even people with time-separable preferences prefer an early resolution of uncertainty, and that preference in fact takes the form studied in the literature.[5] So while a preference for information is often inserted into endowment-economy models by assumption, it is in fact a natural characteristic of a standard model with a consumption/saving choice.

Finally, the results derived here are valuable for providing insight into how an extraordinarily broad range of models with endogenous consumption decisions may be solved exactly. The penultimate section of the paper illustrates representative examples.

1 Solutions to consumption-savings problems

1.1 Constant absolute risk aversion and income risk

Consider an agent with the period utility function $-\alpha^{-1}\exp(-\alpha C_t)$ and time discount factor β. The agent maximizes discounted expected utility,

$$E_t \sum_{j=0}^{\infty} \beta^j \left(-\alpha^{-1}\right) \exp\left(-\alpha C_{t+j}\right) \tag{5}$$

where E_t is the expectation operator conditional on information available on date t. The agent has access to a riskless savings technology and an exogenous income stream, Y_t. Wealth follows

$$W_t = RW_{t-1} + Y_t - C_t \tag{6}$$

The savings technology can be interpreted in a range of ways. For an individual consumer, it can represent a savings account at a bank with a fixed interest rate. For a small open economy, it represents savings at a fixed world interest rate. In a large closed economy, it could be interpreted as an AK production technology (where W_t then represents capital), in which case this becomes an endogenous growth model.

Theorem 1 ***Consumption rule.*** *The consumption rule, C_t^*, that maximizes CARA utility (5) subject to the law of motion for wealth (6) is*

$$C_t^* = (R-1)\, W_{t-1} + Z_t - (R-1)^{-1}\, \alpha^{-1} \log(\beta R) \tag{7}$$

where Z_t follows the recursion

$$Z_t = \left(1 - R^{-1}\right) Y_t - R^{-1}\alpha^{-1} \log E_t \exp\left(-\alpha Z_{t+1}\right) \tag{8}$$

[5] The preference for an early resolution of uncertainty has been commented upon by Kreps and Porteus (1979), Machina (1984), Epstein and Zin (1989), and Ergin and Sarver (2015), but, again, the exact form of the preference was not previously noted.

(See the appendix for derivations of all results). The optimal consumption rule thus takes on a recognizable form. First, the agent consumes the annuity value of financial wealth (or capital), $(R-1)\,W_{t-1}$. Second, the agent consumes the annuity value of the expected income stream, which is encoded by Z_t.[6] Income is not discounted at a simple fixed interest rate in some simple net present value form, though. Rather, the present value of income is defined recursively. Given the form of Z, there is a direct link to generalized recursive preferences:

Result 2 ***Indirect utility.*** *The indirect utility of a CARA agent over income streams is in the class of generalized recursive preferences studied by Epstein and Zin (1989), where the time aggregator is linear and the risk aggregator is CARA with risk aversion α. Specifically, utility over income streams is represented by Z_t.*

This second result follows from the fact that the Euler equation for the agent here implies that $\exp\left(-\alpha C_t^*\right) = E_t \beta R \exp\left(-\alpha C_{t+1}^*\right)$, so that lifetime utility under the optimal consumption policy is proportional to current utility,

$$E_t \sum_{j=0}^{\infty} \beta^j \left(-\alpha^{-1}\right) \exp\left(-\alpha C_{t+j}^*\right) = \frac{-\alpha^{-1}}{1-R^{-1}} \exp\left(-\alpha C_t^*\right) \tag{9}$$

To be clear, the result here is about indirect utility. It says that agents rank possible income streams according to an Epstein–Zin type aggregator. Whereas Epstein–Zin preferences are typically used in reference to consumption streams, here they represent indirect utility over income, Y_t, so Y_t is what enters the recursion (equation (8)).

For a wide class of models, Z_t can be expressed as a linear function of the state of the economy:

Result 3 ***Analytic solutions.*** *If $Y_t = aX_t$, where a and X are conformable vectors and X_t is an affine process in the sense of Duffie, Filipovic, and Schachermayer (2003) (the cumulant generating function of X_{t+j} conditional on X_t is affine in X_t), then Z_t has a solution of the form,*

$$Z_t = zX_t \tag{10}$$

for a vector z if there is a solution to the equation

$$z = \left(1 - R^{-1}\right) a - R^{-1}\alpha^{-1}\varphi\left(-\alpha z\right) \tag{11}$$

where φ is a function that determines how the conditional cumulant-generating function of X_{t+1} depends on X_t

$$\log E_t \exp\left(mX_{t+1}\right) = \varphi\left(m\right) X_t \tag{12}$$

[6] This consumption rule is implicit in results obtained by Caballero (1990), but the explicit form for Z and the link to recursive preferences were not noted there.

While that result may initially appear complicated, the assumption that X is affine is satisfied by a wide range of processes. For example, it holds if Y_t is driven by an ARMA process with i.i.d. innovations. It is also satisfied by the dynamic process for consumption used in the long-run risk model of Bansal and Yaron (2004) or Drechsler and Yaron (2011) and the model of time-varying income risk of Schmidt (2015) (the vast majority of models in the consumption-based asset pricing literature with Epstein–Zin preferences use affine endowment processes). That said, the equation (11) need not have a solution, a situation discussed in Campbell, Giglio, Polk, and Turley (2015) for a related recursion.

Now since a unit of income can be converted directly into a unit of consumption, the marginal utility of a unit of consumption is equivalent to the marginal utility of a unit of income. That means that the pricing kernel may be derived from the indirect utility over income.

Result 4 ***Pricing kernel.*** *The agent's pricing kernel for arbitrary assets (or stochastic discount factor) is*

$$M_{t+1} = \beta \exp\left(-\alpha \Delta C_{t+1}\right) \tag{13}$$

$$= R^{-1} \frac{\exp\left(-\alpha Z_{t+1}\right)}{E_t \exp\left(-\alpha Z_{t+1}\right)} \tag{14}$$

$$= \frac{\partial Z_t / \partial Y_{t+1}}{\partial Z_t / \partial Y_t} \tag{15}$$

where Δ denotes the first-difference operator.

So the fact that the agent's preferences over income streams take the Epstein–Zin recursive form also leads to an alternative representation for the pricing kernel in terms of the present value of income, instead of consumption. Moreover, that pricing kernel can be obtained by differentiating the utility function in the same manner that one typically does to obtain the pricing kernel under Epstein–Zin preferences. This implies that assets can be priced based on their covariance with innovations to the income process, through Z, in addition to consumption.

1.2 Constant relative risk aversion and return risk

Now consider an agent with time-separable constant relative risk aversion preferences. The agent maximizes

$$E_t \sum_{j=0}^{\infty} \exp\left(-\beta\right)^j \frac{C_{t+j}^{1-\gamma}}{1-\gamma} \tag{16}$$

From here on, we denote logs of variables with lower-case letters, e.g. $c_t = \log C_t$. We now model wealth as following the process

$$w_{t+1} = k + \left(1 - \rho^{-1}\right) c_t + \rho^{-1} w_t + \rho^{-1} r_{t+1} \tag{17}$$

where ρ is a constant parameter. The standard motivation for such a law of motion is that it is the Campbell–Shiller (1988) approximation to the return on wealth.

Theorem 5 *The consumption rule, C_t^*, that maximizes CRRA utility (16) subject to the law of motion for wealth (17) is*

$$c_t^* = w_t + \hat{z}_t - \left(1 - \gamma^{-1}\right) r_t + \frac{\beta\gamma^{-1} - k}{\rho^{-1} - 1} \tag{18}$$

where $\hat{z}_t$ follows the recursion

$$\hat{z}_t = \left(1 - \gamma^{-1}\right) r_t - \rho\gamma^{-1} \log E_t \exp\left(-\gamma \hat{z}_{t+1}\right) \tag{19}$$

We thus again obtain a risk-adjusted recursion, now over the return process, r_t. Moreover, as with CARA preferences, if r_t follows an affine process, $\hat{z}_t$ will potentially have an affine solution (again up to an existence condition).

The important difference between the result here and in the previous subsection is that there is not a simple expression for indirect utility. The reason that result is obtained under CARA preferences is that utility is proportional to marginal utility. That is not true for CRRA preferences, though. That also means that we cannot obtain an expression for the pricing kernel by simply differentiating $\hat{z}_t$. However, there is still an insightful result that can be obtained for the pricing kernel.

Result 6 ***Pricing kernel.*** *The pricing kernel follows*

$$M_{t+1} = \exp\left(-\beta - \gamma \Delta c_{t+1}\right) \tag{20}$$

$$= \exp\left(-r_{t+1} - \gamma\left(\hat{z}_{t+1} - G_t\left(\hat{z}_{t+1}\right)\right)\right) \tag{21}$$

where

$$G_t\left(\hat{z}_{t+1}\right) \equiv -\gamma^{-1} \log E_t \exp\left(-\gamma \hat{z}_{t+1}\right) \tag{22}$$

is a certainty equivalent over $\hat{z}_{t+1}$ with risk aversion $\gamma - 1$.

So the pricing kernel again depends on the innovation to $\hat{z}_{t+1}$, but it now also involves the current return on assets, r_{t+1}.

2 The pricing of economic risks

A major goal of the asset pricing literature is to understand the fundamental economic risks that investors are willing to pay to hedge. Do investors demand compensation for exposure to risk associated with shocks to GDP, or inflation, or exchange rates? Recently, Epstein–Zin preferences have become perhaps the dominant modeling paradigm in consumption-based asset pricing literature.

Those preferences are interesting at least partly because they have extremely rich implications for the types of risks that should be priced. They predict that any shock that affects lifetime utility, including fluctuations in expected future growth rates, volatility, and crash risk, should be priced. And there is evidence that those predictions are borne out in the data.[7]

The results in the previous section imply that the pricing of such risks is in fact a much more general prediction of economic models, both with time-separable CARA and CRRA preferences – they do not require Epstein–Zin preferences. This section derives, as simple examples, the risk premia on growth rate and volatility news under CARA preferences. In other words, the fact that we observe a volatility risk premium does not imply that investors have Epstein–Zin preferences (or some other form of non-separability over time). We would expect to observe a risk premium for fluctuations in volatility even if investors had purely time-separable preferences.

Again, it is important to note that the key difference between time-separable utility and Epstein–Zin preferences is that under time-separable utility, the prediction is that stochastic volatility and crash risk in *income* (or returns) is priced, whereas under Epstein–Zin preferences it is stochastic volatility and crash risk in *consumption* that is priced. The claim here is not that the different preference specifications are identical. Rather, it is simply that the deep economic prediction – that time variation in risk is priced – does not require Epstein–Zin preferences.

2.1 Preference for early resolution of uncertainty

Epstein–Zin preferences are often described as generating a preference for an early or late resolution of uncertainty, as compared to time-separable utility, that supposedly induces no preference over the timing of information revelation. And when referring to pure consumption streams, that distinction is accurate. But when agents are able to make a decision based on information, they typically prefer an early resolution of uncertainty, even if they have time separable preferences.

That result is clearly implied by the fact that indirect utility over the income and return processes take on Epstein and Zin's (1989) recursive form. To make the result clear, consider an agent who faces an income process $Y_t = b(L)\varepsilon_t$, where L is the lag operator and b a power series in L. The agent is offered the chance to obtain information revealing the value of ε_{t+1}. We then have the following result

Result 7 *For an agent with CARA preferences and the income process $Y_t = b(L)\varepsilon_t$, where $\varepsilon_t \sim N(0,1)$, the unconditional mean of lifetime utility is*

$$\bar{z} = -\frac{\alpha}{2}R^{-1}\left(1-R^{-1}\right)b\left(R^{-1}\right)^2 \tag{23}$$

If the agent is always to observe ε one period in advance (she is told ε_{t+1} on date t), then average

[7] See, e.g., Coval and Shumway (2001), Campbell and Vuolteenaho (2004), Bansal, Dittmar, and Kiku (2009), Bansal et al. (2013), Campbell et al. (2015), and Dew-Becker and Giglio (2016), among many others.

utility, $\bar{z}^*$, is

$$\bar{z}^* = -\frac{\alpha}{2} R^{-3} \left(1 - R^{-1}\right) b \left(R^{-1}\right)^2 = R^{-2} \bar{z} \tag{24}$$

Since $\bar{z} \leq 0$, $\bar{z}^* \geq \bar{z}$. More generally, if she learns the value of ε_{t+j} on date t, utility, $\bar{z}^{*j}$ becomes

$$\bar{z}^{*(j)} = R^{-2j} \bar{z} \geq \bar{z} \tag{25}$$

This result gives a simple way to see how an agent with time-separable preferences places value on information about the future. $b\left(R^{-1}\right)$ is the net present value of an innovation to income, ε_t, discounted by R. The term $\bar{z}$ is weakly negative and is a penalty due to uncertainty about future income (the inequalities are all strict if $\sigma^2 > 0$). When the agent is given information about future values of the innovation ε, the variance of the news about the NPV of income in each period shrinks (simply because when the news is about income farther in the future, so it is discounted more heavily). That is why the variance penalty in utility, $\bar{z}^{*(j)}$, scales with the squared inverse discount factor.

The preference for early resolution of uncertainty is thus a deep feature of economic behavior. When there is an action that agents may take conditional on information – in this case the choice of how much to consume and save – they can directly benefit from information. While that result was known previously (e.g. Mossin (1969), Dreze and Modigliani (1972), Spence and Zeckhauser (1972), Kreps and Porteus (1979), and Ergin and Sarver (2015)), the results reported here make it analytically formal in an asset pricing context. Specifically, while previous work has made the general point that preferences over income are not time separable and induce a preference for early resolution of uncertainty, this paper is novel for making explicit the connection to a class of preferences widely used in the recent asset pricing literature.

2.2 Pricing of growth rate risk

Bansal and Yaron (2004) show that persistent fluctuations in consumption growth are priced under Epstein–Zin preferences. A similar result applies under time-separable preferences to persistent fluctuations in income.

Suppose income has both persistent and transitory components,

$$Y_t = x_t + \varepsilon_t \tag{26}$$

$$x_t = \phi x_{t-1} + \mu_t \tag{27}$$

It is then straightforward to show that the innovation in the log pricing kernel, $m_{t+1} - E_t m_{t+1}$, follows

$$m_{t+1} - E_t m_{t+1} = -\alpha \left(\frac{1 - R^{-1}}{1 - R^{-1}\phi} \mu_{t+1} + \left(1 - R^{-1}\right) \varepsilon_{t+1} \right) \tag{28}$$

The innovations to the pricing kernel thus depend on both news about current income, through ε_t, and also news about future income, through μ_t. Moreover, shocks to μ_t get a larger risk price since they have larger effects on permanent income.

The basic intuition behind the long-run risk model, that persistent fluctuations in economic growth can receive large risk premia, therefore appears with time-separable utility. Epstein–Zin preferences are only needed if the persistent fluctuations in economic growth are to be measured with consumption, rather than income. But since consumption and income are cointegrated under balanced growth, their long-run properties must ultimately be highly similar (or identical) in any case.

2.3 Pricing of variance risk

The variance risk premium has been widely studied in the recent literature (since Coval and Shumway (2001)). Structural models of the variance risk premium typically involve Epstein–Zin preferences (Drechsler and Yaron (2011), since they directly imply that shocks to consumption volatility are priced. But, again, that is a much more general prediction that also applies under time-separable preferences. Moreover, the empirical literature rarely directly measures variation in consumption volatility over time. Rather, it typically focuses on return volatility (e.g. Drechsler and Yaron (2011) and Campbell et al. (2015)).

As a specific example, suppose income is uncorrelated over time, but that it has stochastic volatility,

$$Y_t = \sigma_{t-1}\varepsilon_t \tag{29}$$

$$\sigma_t^2 = (1-\phi)\,\bar{\sigma}^2 + \phi\sigma_{t-1}^2 + k\omega_t \tag{30}$$

where ε and ω are both standard normals. One can then show that under CARA preferences, the innovation in the log pricing kernel is

$$m_{t+1} - E_t m_{t+1} = -\alpha\left(\left(1-R^{-1}\right)\sigma_t\varepsilon_{t+1} - \frac{\alpha}{2}\frac{R^{-1}\left(1-R^{-1}\right)^2}{1-R^{-1}\phi}k\omega_{t+1}\right) \tag{31}$$

So shocks to volatility, ω_{t+1}, are again priced, in addition to shocks to income itself ($\sigma_{t-1}\varepsilon_t$). In the case where agents have power utility, an analogous derivation shows that fluctuations in the conditional volatility of asset returns are priced.

2.4 The intertemporal CAPM and pricing of higher moments

The two examples above apply to CARA preferences, but more standard CRRA preferences imply similar results. Recall from above that the pricing kernel for power utility with uncertainty about the returns process is

$$M_{t+1} = \exp\left(-\left(\gamma+1\right)\beta - r_{t+1} - \gamma\left(\hat{z}_{t+1} - G_t\left(\hat{z}_{t+1}\right)\right)\right) \tag{32}$$

where

$$\hat{z}_t = \left(1-\gamma^{-1}\right) r_t - \rho\gamma^{-1}\log E_t \exp\left(-\gamma \hat{z}_{t+1}\right) \tag{33}$$

$$G_t\left(\hat{z}_{t+1}\right) \equiv -\gamma^{-1}\log E_t \exp\left(-\gamma \hat{z}_{t+1}\right) \tag{34}$$

This result is a generalization of the intertemporal CAPM studied in Campbell (1993), Campbell and Vuolteenaho (2004), and Campbell et al. (2015). It says that the pricing kernel depends on both current returns, r_{t+1}, and also the innovation in the certainty equivalent over future returns, $\hat{z}_{t+1}$. Campbell (1993) focuses on the case where returns are homoskedastic, while Campbell et al. (2015) assume that they are conditionally Gaussian with volatility that follows a linear process. The result here shows the analysis of the type pursued in those papers actually applies to much more general processes. Specifically, if returns follow an affine process, there will generally be a closed-form linear expression for $\hat{z}_t$ and $G_{t-1}\left(\hat{z}_t\right)$. So a linear pricing model is obtained in much more general settings than models assuming that returns are conditionally Gaussian. As noted by Campbell (1993), equations (20) and (32) are essentially a form of the intertemporal CAPM studied by Merton (1969), in this case allowing for very general specifications for the return process.

As a simple example, suppose returns have a Gaussian component but also Poisson-distributed downward jumps,

$$r_t = \varepsilon_{r,t} - N_t J \tag{35}$$

$$\varepsilon_{r,t} \sim N\left(0,\sigma_r^2\right) \tag{36}$$

$$N_t \sim Poisson\left(\lambda_t\right) \tag{37}$$

$$\lambda_t = \left(1-\phi\right)\bar{\lambda} + \phi\lambda_{t-1} + \varepsilon_{\lambda,t} \tag{38}$$

$$\varepsilon_{\lambda,t} \sim N\left(0,\sigma_\lambda^2\right) \tag{39}$$

One may then show that the innovation in the pricing kernel is

$$m_{t+1} - E_t m_{t+1} = -\gamma r_{t+1} + \frac{\rho}{1-\rho\phi}\left(\exp\left(-\left(\gamma-1\right)J\right)-1\right)\varepsilon_{\lambda,t+1} \tag{40}$$

Variation in jump risk is then priced, and the price of risk is increasing the magnitude of the jumps, μ, and the persistence of jump risk, ϕ.

2.5 Summary

This section shows that the types of shocks to the economy that are expected to be priced in asset markets are qualitatively identical under Epstein–Zin and time-separable preferences. Any shock that affects long-run growth rates, volatility, or the higher moments of future income or market returns may be priced. Obviously the specific quantitative implications depend on the parameterization of the preferences and will vary across calibrations. But the basic intuition turns out to be the same: all agents prefer an early resolution of uncertainty, not just those with Epstein–

Zin preferences.

3 Closed-form solutions

The result in theorem 1 is useful not just for showing what types of economic shocks should be priced. It also points to analytic solutions for a number of difficult problems studied in the finance literature. While the solutions are limited to the CARA setting, they are nevertheless useful for providing clear intuition. This section provides three examples of exact solutions.

3.1 Optimal risk sharing

Suppose there are two agents who must share an income stream that follows the process

$$Y_t = b(L)\varepsilon_t \tag{41}$$

$$\varepsilon_t \sim N(0,1) \tag{42}$$

We consider a social planner's optimization problem, where the objective is to maximize the sum of the utilities of the two agents. We look for optimal solutions in the class of linear models, where one agent is assigned income following $h(L)\varepsilon_t$ and the other receives the remainder, $(b(L)-h(L))\varepsilon_t$. The agents are allowed to freely save and consume. The social planner simply chooses the optimal allocation of income.

In the case of two agents with risk aversion α_1 and α_2, the optimal allocation to agents 1 and 2 can be shown to satisfy the condition

$$h\left(R^{-1}\right) = \frac{\alpha_1^{-1}}{\alpha_1^{-1}+\alpha_2^{-1}} b\left(R^{-1}\right) \tag{43}$$

$$b\left(R^{-1}\right) - h\left(R^{-1}\right) = \frac{\alpha_2^{-1}}{\alpha_1^{-1}+\alpha_2^{-1}} b\left(R^{-1}\right) \tag{44}$$

The riskiness of an income stream is measured by the standard deviations of the innovations to permanent income (discounted at the market interest rate), which are measured by $b\left(R^{-1}\right)$. That risk is then optimally allocated to the agents in proportion to their risk tolerance. This result can easily be extended to a setting with many agents, which case agent j's allocation is

$$\frac{\alpha_j^{-1}}{\sum_k \alpha_k^{-1}} b\left(R^{-1}\right) \tag{45}$$

The allocations are indeterminate except for the restrictions on $h\left(R^{-1}\right)$. That is, there are many polynomials $h(L)$ that can generate any particular value for $h\left(R^{-1}\right)$. Again, what matters for utility is the volatility of shocks to permanent income. The changes in permanent income can come through an immediate permanent step up or down in income, or a change in growth rates over a longer period of time. Agents are indifferent to the two scenarios (holding $h\left(R^{-1}\right)$ constant) because they are able to borrow and save to smooth out any predictable variation in income.

3.2 Volatility trading

The volatility risk premium has been a major focus of recent asset pricing research. A natural question, then, is who we would expect to be trading claims to volatility. That is, the analysis above showed that we should expect a risk premium on volatility, but what patterns would we expect to see in trade in volatility?

Suppose there are two agents, indexed by i, with income processes that are heteroskedastic and may not be traded (such as labor income). For simplicity, assume they are uncorrelated over time and across agents,

$$Y_{i,t} = \sigma_{t-1}\varepsilon_{i,t} \tag{46}$$

$$\sigma_t^2 = 1 + s(L)\,\mu_t \tag{47}$$

The volatility of their income processes is driven by a common factor, σ_t.

So we again consider a social planner who tries to maximize joint utility. In this case, the agents may not share $Y_{i,t}$, but the planner can make transfers conditional on the level of volatility. In other words, this is a setting where income itself is untradeable, but claims to the aggregate volatility process are.

The income of the two agents after the transfer is

$$Y_{1,t} = \sigma_{t-1}\varepsilon_{1,t} + v(L)\,\mu_t \tag{48}$$

$$Y_{2,t} = \sigma_{t-1}\varepsilon_{2,t} - v(L)\,\mu_t \tag{49}$$

where $v(L)$ is a lag polynomial optimally chosen by the social planner. It represents a payment from agent 2 to agent 1.

One may show that the optimal payment, $v(L)$, satisfies the condition

$$v\left(R^{-1}\right) = R^{-1}\frac{1}{2}\left(\alpha_1 - \alpha_2\right)\left(1 - R^{-1}\right)s\left(R^{-1}\right) \tag{50}$$

So, again, the planner's problem depends only on the NPV of the transfers, measured by $v\left(R^{-1}\right)$. The optimal transfer scheme makes payments to the agent who is more risk averse in periods when there are positive shocks to volatility. Intuitively, the marginal utility of the more risk averse agent rises when volatility rises, since she then discounts future income more aggressively (since the certainty equivalent determining Z for her has higher risk aversion). The magnitude of the payments is scaled by the long-run volatility of volatility, $s\left(R^{-1}\right)$. A practical implication of this is that if idiosyncratic income risk varies over time, e.g. due to variation in the layoff rate in the economy, we would expect to see transfers between agents of different risk aversion. If different agents face fluctuations in risk of different size, then there should be a transfer from agents with relatively more stable income risk to those whose risk increases in bad times. That is, even if fluctuations in income themselves are not insurable, fluctuations in income *risk* may be.

4 Conclusion

This paper describes a simple but incredibly general solution to a standard consumption savings problem. The CARA utility function is a widely used benchmark, in particular in models of trade among differentially informed agents (e.g. Grossman and Stiglitz (1980)) and in portfolio choice. The assumption of a simple riskless savings technology is obviously highly restrictive, but also not necessarily too damaging when the object of study is the consumption behavior of individuals.

The results show that the key economic implication of generalized recursive preferences for risk premia – that news about the future state of the economy is priced – is in fact a general feature of models, and not restricted to settings where agents have an *intrinsic* preference for an early resolution of uncertainty. In fact, when agents are able to choose between consuming and saving income, a preference for early resolution of uncertainty is induced. The induced preference for early resolution causes the same economic risks to be priced as when agents have intrinsic preferences for early resolution.

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A Appendix

A.1 CARA preferences

We confirm that the optimal consumption rule is

$$C_t = (R-1)\, W_{t-1} + Z_t - (R-1)^{-1}\, \alpha^{-1} \log(\beta R) \tag{51}$$

where we will solve for Z_t.

The Euler equation is

$$(\beta R)^{-1} = E_t \exp\left(-\alpha\left(C_{t+1} - C_t\right)\right) \tag{52}$$

Given the guess for the consumption rule, wealth follows

$$W_t = W_{t-1} + Y_t - Z_t + (R-1)^{-1}\, \alpha^{-1} \log(\beta R)$$

Which implies that C_{t+1} and consumption growth are

$$\begin{aligned} C_{t+1} &= (R-1)\left(W_{t-1} + Y_t - Z_t\right) + Z_{t+1} - (R-1)^{-1}\, \alpha^{-1} \log(\beta R) + \alpha^{-1} \log(\beta R) && (53) \\ C_{t+1} - C_t &= (R-1)\, Y_t + Z_{t+1} - RZ_t + \alpha^{-1} \log(\beta R) && (54) \end{aligned}$$

Plugging into the Euler equation yields,

$$\begin{aligned} (\beta R)^{-1} &= E_t \exp\left(-\alpha\left((R-1)\, Y_t + Z_{t+1} - RZ_t + \alpha^{-1} \log(\beta R)\right)\right) && (55) \\ \alpha^{-1} \log(\beta R) &= -\alpha^{-1} \log E_t \exp\left(-\alpha\left((R-1)\, Y_t - RZ_t + Z_{t+1} + \alpha^{-1} \log(\beta R)\right)\right) && (56) \end{aligned}$$

$$Z_t = \left(1 - R^{-1}\right) Y_t - R^{-1}\alpha^{-1} \log E_t \exp\left(-\alpha Z_{t+1}\right) \tag{57}$$

This then directly yields the consumption function. That is, the result is

$$C_t = (R-1) W_{t-1} + Z_t - (R-1)^{-1} \alpha^{-1} \log(\beta R) \tag{58}$$

$$Z_t = \left(1 - R^{-1}\right) Y_t - \frac{1}{\alpha} R^{-1} \log E_t \exp\left(-\alpha Z_{t+1}\right) \tag{59}$$

The Euler equation implies

$$\exp(-\alpha C_t) = \beta R E_t \exp(-\alpha C_{t+1}) \tag{60}$$

$$\exp(-\alpha C_{t+1}) = \beta R E_{t+1} \exp(-\alpha C_{t+2}) \tag{61}$$

$$\exp(-\alpha C_t) = \beta R E_t \left[\beta R E_{t+1} \exp(-\alpha C_{t+2})\right] \tag{62}$$

$$= (\beta R)^2 E_t \exp(-\alpha C_{t+2}) \tag{63}$$

Iterating forward then implies that $\exp(-\alpha C_t) = (\beta R)^j E_t \exp(-\alpha C_{t+j})$. So then lifetime utility is

$$V_t^C = -\alpha^{-1} \sum_{j=0}^{\infty} \beta^j E_t \exp(-\alpha C_{t+j}) \tag{64}$$

$$= -\alpha^{-1} \sum_{j=0}^{\infty} \beta^j \frac{\exp(-\alpha C_t)}{(\beta R)^j} \tag{65}$$

$$= \frac{-\alpha^{-1}}{1 - R^{-1}} \exp(-\alpha C_t) \tag{66}$$

Plugging in the consumption function then yields

$$V_t^C = \frac{-\alpha^{-1}}{1 - R^{-1}} \exp\left(-\alpha \left((R-1) W_{t-1} + Z_t - (R-1)^{-1} \alpha^{-1} \log(\beta R)\right)\right) \tag{67}$$

But this is just an increasing transformation of Z_t itself, which implies that Z_t also measures lifetime utility across different income paths.

Z_t is Epstein–Zin utility over the income stream, where the EIS is infinity and the certainty equivalent uses CARA utility instead of power utility (which fits into the framework of Epstein and Zin (1989)). That is, the preference ordering over income streams is represented by a particular form of Epstein–Zin preferences over the income stream.

The pricing kernel is

$$
\begin{aligned}
M_{t+1} &= \beta \frac{\exp\left(-\alpha C_{t+1}\right)}{\exp\left(-\alpha C_t\right)} && (68)\\
&= R^{-1} \frac{\exp\left(-\alpha C_{t+1}\right)}{E_t \exp\left(-\alpha C_{t+1}\right)} && (69)\\
&= R^{-1} \frac{\exp\left(-\alpha\left(\left(R-1\right) W_t + Z_{t+1} - \alpha^{-1}\frac{\log \beta R}{R-1}\right)\right)}{E_t \exp\left(-\alpha\left(\left(R-1\right) W_t + Z_{t+1} - \alpha^{-1}\frac{\log \beta R}{R-1}\right)\right)} && (70)\\
&= R^{-1} \frac{\exp\left(-\alpha Z_{t+1}\right)}{E_t \exp\left(-\alpha Z_{t+1}\right)} && (71)
\end{aligned}
$$

Finally, to obtain the expression for Z in the affine case, we have

$$
\begin{aligned}
zX_t &= \left(1-R^{-1}\right) aX_t - \alpha^{-1}R^{-1} \log E_t \exp\left(-\alpha z X_{t+1}\right) && (72)\\
zX_t &= \left(1-R^{-1}\right) aX_t - \alpha^{-1}R^{-1} \varphi\left(-\alpha z\right) X_t && (73)
\end{aligned}
$$

The equation in the text comes from simply matching the coefficients on the two sides of that equation, since it must hold for all X.

A.2 CRRA preferences

The investor's Euler equation is

$$
1 = E_t\left[\exp\left(-\beta + r_{t+1} - \gamma \Delta c_{t+1}\right)\right] \tag{74}
$$

Now we guess that

$$
c_t = \bar{b} + b_w w_t + z_t \tag{75}
$$

$$
w_{t+1} = k + \left(1-\rho^{-1}\right) c_t + \rho^{-1} w_t + r_{t+1} \tag{76}
$$

We have

$$
\begin{aligned}
c_{t+1} &= \bar{b} + b_w w_{t+1} + z_{t+1} = \bar{b} + b_w\left(r_{t+1} + k + \left(1-\rho^{-1}\right) c_t + \rho^{-1} w_t\right) + z_{t+1} && (77)\\
\Delta c_{t+1} &= \bar{b} + b_w\left(r_{t+1} + k + \rho^{-1} w_t\right) + z_{t+1} + \left(b_w\left(1-\rho^{-1}\right) - 1\right)\left(\bar{b} + b_w w_t + z_t\right) && (78)\\
&= b_w r_{t+1} + z_{t+1} + b_w k + b_w\left(1-\rho^{-1}\right)\bar{b} + b_w\left(\rho^{-1} + b_w\left(1-\rho^{-1}\right) - 1\right) w_t \\
&\quad + \left(b_w\left(1-\rho^{-1}\right) - 1\right) z_t && (79)
\end{aligned}
$$

Inserting into the Euler equation,

$$0 = -\gamma^{-1}\log E_t \exp\left(-\beta + r_{t+1} - \gamma\left(\begin{array}{c} b_w r_{t+1} + z_{t+1} + b_w k + b_w\left(1-\rho^{-1}\right)\bar{b} \\ +b_w\left(\rho^{-1} + b_w\left(1-\rho^{-1}\right) - 1\right) w_t + \left(b_w\left(1-\rho^{-1}\right) - 1\right) z_t \end{array}\right)\right) \tag{80}$$

$$\begin{aligned} 0 &= -\gamma^{-1}\log E_t \exp\left(r_{t+1} - \gamma\left(b_w r_{t+1} + z_{t+1}\right)\right) \\ &\quad +\frac{\beta}{\gamma} + b_w k + b_w\left(1-\rho^{-1}\right)\bar{b} \\ &\quad +b_w\left(\rho^{-1} + b_w\left(1-\rho^{-1}\right) - 1\right) w_t + \left(b_w\left(1-\rho^{-1}\right) - 1\right) z_t \end{aligned} \tag{81}$$

Matching coefficients,

$$w_t : b_w = 1 \tag{82}$$

$$\text{constant} : b_w k + b_w\left(1-\rho^{-1}\right)\bar{b} = -\frac{\beta}{\gamma} \tag{83}$$

The recursion for z is

$$\rho^{-1} z_t = -\gamma^{-1}\log E_t \exp\left(\left(1-\gamma\right) r_{t+1} - \gamma z_{t+1}\right) \tag{84}$$

Now define $\hat{z}_t$ such that

$$\gamma \hat{z}_t \equiv \gamma z_t + \left(\gamma - 1\right) r_t \tag{85}$$

We then have

$$\hat{z}_t = \left(1-\gamma^{-1}\right) r_t - \rho\gamma^{-1}\log E_t \exp\left(-\gamma \hat{z}_{t+1}\right) \tag{86}$$

Finally, then, inserting this into the consumption function,

$$c_t = \bar{b} + b_w w_t + \hat{z}_t - \left(1-\gamma^{-1}\right) r_t \tag{87}$$

$$\begin{aligned} \Delta c_{t+1} &= r_{t+1} + z_{t+1} - \frac{\beta}{\gamma} + \gamma^{-1}\log E_t \exp\left(\left(1-\gamma\right) r_{t+1} - \gamma z_{t+1}\right) \\ -\gamma \Delta c_{t+1} &= -\gamma \hat{z}_{t+1} - r_{t+1} + \beta - \log E_t \exp\left(-\gamma \hat{z}_{t+1}\right) \end{aligned} \tag{88}$$

The pricing kernel is obtained from

$$\begin{aligned} -\gamma \Delta c_{t+1} &= -\gamma \hat{z}_{t+1} - r_{t+1} - \log E_t \exp\left(-\gamma \hat{z}_{t+1}\right) + \beta && (89) \\ &= -r_{t+1} - \gamma\left(\hat{z}_{t+1} - \left(-\gamma^{-1}\right)\log E_t \exp\left(-\gamma \hat{z}_{t+1}\right)\right) + \beta && (90) \\ &= -r_{t+1} - \gamma\left(\hat{z}_{t+1} - G\left(\hat{z}_{t+1}\right)\right) + \beta && (91) \end{aligned}$$

where

$$G\left(\hat{z}_{t+1}\right) \equiv -\gamma^{-1}\log E_t \exp\left(-\gamma \hat{z}_{t+1}\right) \tag{92}$$

A.3 Preference for an early resolution of uncertainty

We guess that $Z_t = \bar{z} + z(L)\varepsilon_t$. We have

$$\bar{z} + z(L)\varepsilon_t = \left(1 - R^{-1}\right) b(L)\varepsilon_t - R^{-1}\alpha^{-1}\log E_t \exp\left(-\alpha\left(\bar{z} + z(L)\varepsilon_{t+1}\right)\right) \tag{93}$$

$$= \left(1 - R^{-1}\right) b(L)\varepsilon_t + R^{-1}\left(\bar{z} + \sum_{j=1}^{\infty} z_j \varepsilon_{t+1-j}\right) - \frac{\alpha}{2}R^{-1}z_0^2 \tag{94}$$

Matching coefficients yields

$$z_j = \left(1 - R^{-1}\right) b_j + R^{-1} z_{j+1} \tag{95}$$

$$z_0 = \left(1 - R^{-1}\right) b\left(R^{-1}\right) \tag{96}$$

$$\bar{z} = -\frac{\alpha}{2}R^{-1}\left(1 - R^{-1}\right) b\left(R^{-1}\right)^2 \tag{97}$$

Now when the agent learns about the shock ε a period in advance, it is as though there is an alternative income process, $Y_t = b(L)\varepsilon_{t-1}$. that immediately implies that $\bar{z}$ under the alternative income process is simply R^{-2} multiplied by $\bar{z}$ for the original process.

A.4 Pricing of growth rate risk

We guess that $Z_t = \bar{z} + \left(1 - R^{-1}\right)\varepsilon_t + z_x x_t$. We then have

$$\bar{z} + \left(1 - R^{-1}\right)\varepsilon_t + z_x x_t = \left(1 - R^{-1}\right)(x_t + \varepsilon_t) \tag{98}$$

$$-R^{-1}\alpha^{-1}\log E_t \exp\left(-\alpha\left(\bar{z} + \left(1 - R^{-1}\right)\varepsilon_{t+1} + z_x x_{t+1}\right)\right) \tag{99}$$

$$= \left(1 - R^{-1}\right)(x_t + \varepsilon_t) + R^{-1}\left(\bar{z} + \phi x_t\right) \tag{100}$$

$$-R^{-1}\frac{\alpha}{2}\left(\left(1 - R^{-1}\right)^2 \sigma_\varepsilon^2 + z_x^2 \sigma_\mu^2\right) \tag{101}$$

which implies

$$z_x = \left(1 - R^{-1}\right) + R^{-1}\phi z_x \tag{102}$$

$$= \frac{1 - R^{-1}}{1 - R^{-1}\phi} \tag{103}$$

That immediately yields the sensitivity of the log pricing kernel to the innovations to ε and μ.

A.5 Pricing of variance risk

$$Y_t = \sigma_{t-1}\varepsilon_t \tag{104}$$

$$\sigma_t^2 = (1 - \phi)\bar{\sigma}^2 + \phi\sigma_{t-1}^2 + k\omega_t \tag{105}$$

We now guess that $Z_t = \bar{z} + \left(1 - R^{-1}\right) Y_t + z_\sigma \sigma_t^2$

$$
\begin{aligned}
Z_t &= \left(1 - R^{-1}\right) Y_t && (106)\\
&\quad - R^{-1}\alpha^{-1} \log E_t \exp\left(-\alpha\left(\bar{z} + \left(1 - R^{-1}\right) Y_{t+1} + z_\sigma \sigma_{t+1}^2\right)\right) && (107)\\
\bar{z} + z_\sigma \sigma_t^2 &= R^{-1}\left(\bar{z} + z_\sigma \phi \sigma_t^2 + z_\sigma \left(1-\phi\right)\bar{\sigma}^2\right) - R^{-1}\frac{\alpha}{2}\left(\left(1 - R^{-1}\right)^2 \sigma_t^2 + z_\sigma^2 k^2\right) && (108)
\end{aligned}
$$

Matching coefficients,

$$z_\sigma = -R^{-1}\frac{\alpha}{2}\frac{\left(1 - R^{-1}\right)^2}{1 - R^{-1}\phi} \tag{109}$$

which is the price of risk from the text.

A.6 Pricing of disaster risk

The recursion for $\hat{z}$ is

$$\hat{z}_t = \left(1 - \gamma^{-1}\right) r_t - \rho\gamma^{-1} \log E_t \exp\left(-\gamma \hat{z}_{t+1}\right) \tag{110}$$

we guess that $\hat{z}$ takes the form $\bar{z} + z_\lambda \lambda_t + \left(1 - \gamma^{-1}\right) r_t$

$$
\begin{aligned}
\bar{z} + z_\lambda \lambda_t + \left(1 - \gamma^{-1}\right) r_t &= \left(1 - \gamma^{-1}\right) r_t - \rho\gamma^{-1} \log E_t \exp\left(-\gamma\left(\bar{z} + z_\lambda \lambda_{t+1} + \left(1 - \gamma^{-1}\right) r_{t+1}\right)\right) && (111)\\
\bar{z} + z_\lambda \lambda_t &= \rho\left(\bar{z} + z_\lambda\left(\left(1-\phi\right)\bar{\lambda} + \phi\lambda_t\right)\right) - \rho\frac{\gamma}{2}\left(z_\lambda^2 \sigma_\lambda^2 + \left(1 - \gamma^{-1}\right)^2 \sigma_r^2\right)\\
&\quad - \rho\gamma^{-1}\lambda_t\left(\exp\left(-\gamma\left(1 - \gamma^{-1}\right) J\right) - 1\right) && (112)
\end{aligned}
$$

Matching coefficients,

$$
\begin{aligned}
z_\lambda &= -\frac{\rho}{1 - \rho\phi}\gamma^{-1}\left(\exp\left(-\gamma\left(1 - \gamma^{-1}\right) J\right) - 1\right) && (113)\\
\bar{z} &= \frac{1}{1-\rho}\left(\rho z_\lambda \left(1-\phi\right)\bar{\lambda} - \rho\frac{\gamma}{2}\left(z_\lambda^2 \sigma_\lambda^2 + \left(1 - \gamma^{-1}\right)^2 \sigma_r^2\right)\right) && (114)
\end{aligned}
$$

$$
\begin{aligned}
-\gamma^{-1} \log E_t \exp\left(-\gamma \hat{z}_{t+1}\right) &= \bar{z} + z_\lambda\left(\left(1-\phi\right)\bar{\lambda} + \phi\lambda_t\right) - \frac{\gamma}{2}\left(z_\lambda^2 \sigma_\lambda^2 + \left(1 - \gamma^{-1}\right)^2 \sigma_r^2\right)\\
&\quad - \gamma^{-1}\lambda_t\left(\exp\left(-\gamma\left(1 - \gamma^{-1}\right) J\right) - 1\right) && (115)\\
\hat{z}_{t+1} - G_t\left(\hat{z}_{t+1}\right) &= z_\lambda \varepsilon_{\lambda,t+1} + \left(1 - \gamma^{-1}\right) r_{t+1} + \frac{\gamma}{2}\left(z_\lambda^2 \sigma_\lambda^2 + \left(1 - \gamma^{-1}\right)^2 \sigma_r^2\right)\\
&\quad + \gamma^{-1}\lambda_t\left(\exp\left(-\gamma\left(1 - \gamma^{-1}\right) J\right) - 1\right) && (116)
\end{aligned}
$$

$$\Delta E_{t+1} \log M_{t+1} = -\gamma r_{t+1} - \gamma z_\lambda \varepsilon_{\lambda,t+1} \tag{117}$$

A.7 Optimal risk sharing

the optimal risk sharing results follow from the result above that when income is $h(L)\varepsilon_t$, then lifetime utility when the history of shocks is set to zero is

$$\bar{z}_1 = -R^{-1}\frac{\alpha_1}{2}\left(1-R^{-1}\right)h\left(R^{-1}\right)^2 \tag{118}$$

For agent 2, utility is then

$$\bar{z}_2 = -R^{-1}\frac{\alpha_2}{2}\left(1-R^{-1}\right)\left(b\left(R^{-1}\right)-h\left(R^{-1}\right)\right)^2 \tag{119}$$

$h\left(R^{-1}\right)$ is obtained by minimizing $\bar{z}_1+\bar{z}_2$.

A.8 Volatility trading

We guess that $Z_t = \bar{z} + z(L)\mu_t + \left(1-R^{-1}\right)\sigma_{t-1}\varepsilon_{i,t}$. The recursion for lifetime utility is,

$$Z_t = \left(1-R^{-1}\right)\sigma_{t-1}\varepsilon_{i,t} + \left(1-R^{-1}\right)v(L)\mu_t + R^{-1}\left(\bar{z}+\sum_{j=1}^{\infty}z_j\mu_{t+1-j}\right) \tag{120}$$

$$-R^{-1}\frac{\alpha}{2}\left(1-R^{-1}\right)^2\left(1+s(L)\mu_t\right) - R^{-1}\frac{\alpha}{2}z_0^2 \tag{121}$$

Matching coefficients,

$$z_j = R^{-1}z_{j+1} - R^{-1}\frac{\alpha}{2}\left(1-R^{-1}\right)^2 s_j + \left(1-R^{-1}\right)v_j \tag{122}$$

$$z_0 = -R^{-1}\frac{\alpha}{2}\left(1-R^{-1}\right)^2 s\left(R^{-1}\right) + \left(1-R^{-1}\right)v\left(R^{-1}\right) \tag{123}$$

$$\bar{z} = \frac{-R^{-1}}{1-R^{-1}}\frac{\alpha}{2}\left(\left(1-R^{-1}\right)^2 + z_0^2\right) \tag{124}$$

The planner's objective is to maximize

$$V = -\frac{\alpha_1}{2}\left(-R^{-1}\frac{\alpha}{2}\left(1-R^{-1}\right)^2 s\left(R^{-1}\right) + \left(1-R^{-1}\right)v\left(R^{-1}\right)\right)^2 \tag{125}$$

$$-\frac{\alpha_2}{2}\left(-R^{-1}\frac{\alpha}{2}\left(1-R^{-1}\right)^2 s\left(R^{-1}\right) - \left(1-R^{-1}\right)v\left(R^{-1}\right)\right)^2 \tag{126}$$

The first-order condition for $v\left(R^{-1}\right)$ is

$$0 = -\alpha_1\left(-R^{-1}\frac{\alpha}{2}\left(1-R^{-1}\right)^2 s\left(R^{-1}\right) + \left(1-R^{-1}\right)v\left(R^{-1}\right)\right)\left(1-R^{-1}\right) \tag{127}$$

$$+\alpha_2\left(-R^{-1}\frac{\alpha}{2}\left(1-R^{-1}\right)^2 s\left(R^{-1}\right) - \left(1-R^{-1}\right)v\left(R^{-1}\right)\right)\left(1-R^{-1}\right) \tag{128}$$

which then yields

$$\alpha_2 \left(-R^{-1}\frac{\alpha}{2}\left(1-R^{-1}\right)s - v\right) = \alpha_1 \left(-R^{-1}\frac{\alpha}{2}\left(1-R^{-1}\right)s + v\right) \tag{129}$$

$$R^{-1}\left(\frac{\alpha_1^2}{2}-\frac{\alpha_2^2}{2}\right)\left(1-R^{-1}\right)s = \left(\alpha_1+\alpha_2\right)v \tag{130}$$

$$R^{-1}\frac{1}{2}\left(\alpha_1-\alpha_2\right)\left(1-R^{-1}\right)s = v \tag{131}$$

where v is a short-hand for $v\left(R^{-1}\right)$ and s for $s\left(R^{-1}\right)$.