

The effects of restricting high-frequency investment in a noisy rational expectations model*

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Abstract

This paper develops a simple benchmark result for the effects that restrictions on investment strategies have on price informativeness and return volatility. The analysis requires a model with two key characteristics: investors hold portfolios with exposures to fundamentals that vary across dates, and they endogenously choose to learn different amounts about fundamentals at different horizons in the future. Those features can be captured in a simple noisy rational expectations model of a futures market in which trade and information acquisition happen on a single date. In our frictionless benchmark, restricting investors from holding portfolios exposed to variation in fundamentals at some set of frequencies has no effect on the behavior of asset prices and returns at other frequencies, nor does it increase the utility or profits of investors who focus on the unrestricted frequencies. Similar results are obtained for a quadratic tax on changes in investors' positions.

The goal of this paper is to understand how restrictions on the strategies that investors may follow affect their information acquisition decisions and hence the efficiency of prices. We are specifically interested in the effect of restrictions on the frequency with which investment strategies may vary. While there is some recent work on the consequences of various limits on information gathering ability,¹ and there have been empirical analyses of high-frequency traders,² we are not aware of any other work that directly studies the effects of restrictions on high- and low-frequency strategies.³ The question strikes us as being of clear objective importance given the interest of investors and policy makers in such restrictions.

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¹See Banerjee and Green (2015), Goldstein and Yang (2015), Dávila and Parlato (2016), and Farboodi and Veldkamp (2017). Farboodi and Veldkamp (2017) study the effect of changes in information acquisition technologies over time.

²E.g. Brogaard, Hendershott, and Riordan (2014), Hendershott and Riordan (2013) and Zhang (2010).

³Cartea and Penalva (2012) are perhaps closest. They study a model in which there are exogenously given high- and low-frequency traders and examine how the high-frequency traders affect the prices received by the others.

In order to analyze the effects of frequency-based restrictions on investment strategies, we require a model with two key characteristics: investors must have a meaningful choice about types of information to acquire, and they must also be able to choose among investment strategies with exposures that differ across horizons or frequencies.⁴ The paper develops a noisy rational expectations equilibrium model that has precisely those characteristics and remains highly tractable.

More specifically, we study a setting in which investors trade, on a single date, claims on future values of a dividend process. These can be thought of as equity or dividend futures (which are traded in the real world; see Binsbergen, Brandt, and Koijen (2012) and Binsbergen and Koijen (2017)). Variation in a portfolio’s weights across maturities represents variation in the exposure that the portfolio has to fundamentals at different horizons. Investors in the model are also able to acquire information about the future realizations of fundamentals across horizons. All trade happens on a single date, so the model is not fully dynamic, but it has the two features that we desire: exposures and information acquisition choices that may both vary across horizons.

The first contribution of the paper is to show that the type of futures market we study has a general solution that applies when fundamentals follow arbitrary ARMA processes. As a technical matter, the solution is simply a case of what is solved by Admati (1985), and it is easily obtained by hand. Admati (1985) focuses on the fact that the equilibrium in general multi-asset models such as ours is generally extremely difficult to interpret, and usual intuitions, such as increases in supply or reductions in demand reducing prices, fail to hold. In the case of a futures market, though, we show that there is a natural frequency transformation that eliminates those difficulties.⁵ The equilibrium turns out to be completely separable across frequencies.

We use the futures market equilibrium to study the effect of restrictions on investment policies on price informativeness. A natural constraint that might be imposed on a portfolio manager, either by their investors or by a regulator, is a restriction on how rapidly their exposures can vary across dates. At one extreme are index funds, which are forced to have essentially fixed exposures. Towards the other extreme are trading desks, which are sometimes required to have risk exposures of zero at the end of each trading day, but may still have risk exposure during the day (e.g. Brock and Kleidon (1992) and Menkveld (2013)). More concretely related to our setting with futures markets, some managers are restricted from holding, for example, exposure to “level” or “slope” factors. In the context of our stylized model, the specific investment restrictions say that investors may not hold portfolios of futures whose weights vary across maturities at specified frequencies. A restriction on high-frequency investment, then, means that investors may not hold portfolios whose weights rapidly change signs across maturities.

Our main result is that price informativeness and liquidity are reduced at frequencies targeted by

⁴Shleifer and Vishny (1990) provide an early analysis of a choice by traders to focus on short- versus long-term projects, while Goldstein and Yang (2015) provide a recent study with a choice of different types of signals to learn about. Farboodi and Veldkamp (2017) study a model in which investors can learn whether to learn about fundamentals or demand.

⁵For other related work on frequency transformations, see Bandi and Tamoni (2014), Bernhardt, Seiler, and Taub (2010), Chincó and Ye (2017), Chaudhuri and Lo (2016), Dew-Becker and Giglio (2016), and Kasa, Walker, and Whiteman (2013).

such a policy, but not at any others. The model features endogenous information acquisition, as in Goldstein and Yang (2015) and Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016), and at the restricted frequencies, investors have no incentive to acquire information, making prices completely uninformative. But investors continue to obtain information at unrestricted frequencies, meaning that those prices remain equally informative as without the policy.

In the time domain, the consequence of any restriction on trade is to make prices less informative about fundamentals on each individual date. The effects differ, though, for sums of fundamentals over time. As a specific example, consider a policy that discourages investors from holding portfolios with exposures that change rapidly across time periods (i.e. a restriction on high-frequency investment). We show that such a policy increases high-frequency mean-reverting noise in prices. Inference for moving averages of prices therefore is inhibited less, since the high-frequency noise in prices averages out over time. More specifically, the reduction of the informativeness of prices following a restriction on high-frequency investment is smaller for averages of fundamentals over many dates than for fundamentals on a single date.

On a more intuitive level, our aim is to understand how restrictions on trade at different frequencies affect price efficiency. Our results suggest that restricting high-frequency trade may reduce efficiency and liquidity at high frequencies without having adverse effects at lower frequencies. We view our model as a neoclassical benchmark. It does not feature frictions in trade, irrationality, or differences in technology across investors. What it therefore shows is that in order for restrictions on high-frequency investment to have effects at lower frequencies, one must add to the model another friction, or argue that the dynamics that we are not able to model here cause the results to change.

The main drawback of the model that we study is that it is not fully dynamic, meaning that we cannot directly study restrictions on trade, but rather study analogous restrictions on investment policies. There is a literature on dynamic trade, but the extant models do not generate our two desired choices for investors of information acquisition and investment exposures that can differ across horizons. The two main difficulties in solving models of dynamic trade are the private dynamic portfolio choice problem (which typically does not have closed-form solutions; see Chacko and Viceira (2005) for a discussion) and the infinite-regress problem of Townsend (1983). There is work that has made substantial progress in solving the infinite regress problem, but those models assume that investors have only single-period objectives and they do not allow for a choice of information across horizons.⁶ Recent work also examines dynamic models with strategic trade (with similar restrictions regarding horizons), whereas here we study a fully competitive setting in which all investors are price takers.⁷ Relative to the highly sophisticated work on dynamic trade available in the literature, our goal is to take a well understood and highly tractable framework that is used in leading models of information acquisition (e.g. Kacperczyk, Van Nieuwerburgh, and

⁶See Makarov and Rytchkov (2012), Kasa, Walker, and Whiteman (2013), and Rondina and Walker (2017).

⁷For models of dynamic strategic trade, see Vayanos (1999, 2001), Ostrovsky (2012), Banerjee and Breon-Driscoll (2016), Foucault, Hombert, and Rosu (2016), Du and Zhu (2017), and Dugast and Foucault (2017).

Veldkamp (2016)) and study its implications for the effects of investment restrictions.

1 The model

1.1 Market structure

Time is denoted by $t \in \{-1, 0, 1, \dots, T\}$, with T even, and we will focus on cases in which T may be treated as large. There is a fundamentals process D_t , on which investors trade forward contracts, with realizations on all dates except -1 and 0 . The time series process is stacked into a vector $D \equiv [D_1, D_2, \dots, D_T]'$ (versions of variables without time subscripts denote vectors) and is unconditionally distributed as

$$D \sim N(0, \Sigma_D). \tag{1}$$

We assume the fundamentals process is stationary (similar results are obtained if a transformation of D_t is stationary), meaning that it has constant unconditional autocovariances. Stationarity implies that Σ_D is constant along its diagonals, and we further assume that the eigenvalues of Σ_D are finite and bounded away from zero.

There is a set of futures claims on realizations of the fundamental. When we say that the model features a choice of investment across dates, we mean that investors will choose portfolio allocations across the futures contracts, which then yield exposures to the realization of fundamentals on different dates in the future. In order to keep prices from being fully revealing, we assume there is an exogenous supply of each futures contract, in a vector \tilde{Z} , which is unconditionally distributed as

$$\tilde{Z} = kP + Z \tag{2}$$

$$Z \sim N(0, \Sigma_Z). \tag{3}$$

\tilde{Z} may be thought of as either exogenous liquidity demand or unsophisticated trading. The time series process for supply has the same stationarity restrictions as D . \tilde{Z}_t also depends on P_t , which is the price of a claim to fundamentals on date t (and P is the vector of those prices). The scalar $k \geq 0$ represents the sensitivity of supply to prices. When prices are higher, a larger number of the liquidity traders would like to sell. While $k = 0$ is commonly assumed in the literature, $k > 0$ allows us to solve the model even when the sophisticated investors do not provide any liquidity.

A concrete example of a process D_t is the price of crude oil: oil prices follow some stochastic process and investors trade futures on oil at many maturities. D_t can also be interpreted as the dividend on a stock, in which case the futures would be claims on dividends on individual dates. Dividend futures are in fact traded (see Binsbergen and Koijen (2017)). While the concept of a futures market on the fundamentals will be a useful analytic tool, we can also obviously price portfolios of futures. Equity, for example, is a claim to the stream of fundamentals over time. Holding any given combination of futures claims on the fundamental is equivalent to holding futures

contracts on equity claims.

1.2 Information structure

There is a unit mass of “sophisticated” investors indexed by $i \in [0, 1]$. The realization of the time series of fundamentals, $\{D_t\}_{t=1}^T$, can be thought of as a single draw from a multivariate normal distribution. Investors are able to acquire signals about that realization. The signals are a collection $\{Y_{i,t}\}_{t=1}^T$ observed on date 0 with

$$Y_{i,t} = D_t + \varepsilon_{i,t}, \varepsilon_i \sim N(0, \Sigma_i), \quad (4)$$

where Σ_i^{-1} is investor i 's signal precision matrix. Through $Y_{i,t}$, investors can learn about fundamentals on all dates between 1 and T . $\varepsilon_{i,t}$ is a stationary error process in the sense that $\text{cov}(\varepsilon_{i,t}, \varepsilon_{i,t+j})$ depends on j but not t . All dates are thus equally difficult to learn about. The stationarity assumption is imposed so that no particular date is given special prominence in the model. Investors must choose an information policy that treats all dates symmetrically, and they are not allowed to choose to learn about a single date. This assumption yields a useful symmetry between fundamentals, supply, and the signals, in that they are all assumed to be stationary processes.

The signal structure generates one of our desired model features, which is that investors can choose to learn about fundamentals across different dates in the future. Prices may then be more or less informative about fundamentals on different combinations of dates. The serial correlation properties of the errors in the signals determine how useful they are for forecasting moving averages or differences in fundamentals over time. When the errors are positively correlated across dates, the signals are relatively less useful for forecasting trends in fundamentals since the errors also have persistent trends. Conversely, when errors are negatively correlated across dates, the signals are less useful for forecasting transitory variation and provide more accurate information about moving averages.

1.3 Investment objective

On date 0, there is a market for forward claims on fundamentals on all dates in the future. Investor i 's demand for a date- t forward conditional on the set of prices and signals is denoted $Q_{i,t}$. Investors have mean-variance utility over cumulative excess returns:

$$U_{0,i} = \max_{\{Q_{i,t}\}} E_{0,i} \left[T^{-1} \sum_{t=1}^T Q_{i,t} (D_t - P_t) \right] - \frac{1}{2} (\rho T)^{-1} \text{Var}_{0,i} \left[\sum_{t=1}^T Q_{i,t} (D_t - P_t) \right], \quad (5)$$

where $E_{0,i}$ is the expectation operator conditional on agent i 's date-0 information set, $\{P, Y_i\}$. $\text{Var}_{0,i}$ is the variance operator conditional on $\{P, Y_i\}$. ρ is risk-bearing capacity per unit of time. Investors acquire exposures to fundamentals by buying a portfolio of futures contracts on date 0. They then hold that portfolio to maturity and have utility over the mean and variance of the

cumulative excess return of the portfolio.

We interpret the objective as representing a target that of an institutional investor. Rather than aiming to maximize the discounted sum of returns, as a person who consumes out of wealth might, the investors maximize a measure of their performance. The important characteristic of (5) is that it yields a stationary problem in the sense that there is no discounting to make returns in some periods more important than others. Finally, note that all investors have the same investment horizon. Appendix C shows that the horizon has no effect on information choices in the model.

The assumption that signals are acquired and trade occurs on date 0 is obviously restrictive. In general settings there is no known closed-form solution to even the partial-equilibrium dynamic portfolio choice problem, let alone to the full market equilibrium (the frequency-domain solutions to the infinite regress problem, such as Kasa, Walker, and Whiteman (2013) and Makarov and Rytchkov (2012), restrict preferences to avoid the dynamic portfolio problem). Moreover, allowing agents to obtain signals on each date yields a highly nontrivial updating problem. We therefore use a relatively minimal static model. The model nevertheless has the two characteristics that we stated we desire in the introduction: it allows for investment strategies that place different weight on fundamentals on different dates in the future, and it allows investors to make a choice about how precise their signals are for different types of fluctuations in fundamentals.

1.4 Asset market equilibrium in the time domain

We begin by solving for the market equilibrium on date 0 that takes the agents' signal precisions, Σ_i^{-1} , as given. The Σ_i^{-1} are chosen on date -1, and that optimization (which will drive price informativeness) is discussed below. The time-0 asset market equilibrium has a standard definition.

Definition 1 *For any given set of individual precisions $\{\Sigma_i\}_{i \in [0,1]}$, an asset market equilibrium is a set of demand functions, $\{Q_i(P, Y_i)\}_{i \in [0,1]}$, and a price vector P , such that investors maximize utility and all markets clear: $\int_i Q_{i,t} di = \tilde{Z}_t$ for all $t \geq 1$.*

Investors submit demand curves for each futures contract to a Walrasian auctioneer who selects equilibrium prices to clear all markets.

The structure of the time-0 equilibrium is mathematically that of Admati (1985), who studies investment across a set of assets that might represent stocks in different companies, and the solution from that paper applies directly here (though with the minor difference that supply is also a function of prices). Here we are considering investment across a set of futures contracts that represent claims on some fundamentals process across different dates. We simply rotate the Admati (1985) structure from a cross-section to a time series.

Admati's (1985) solution for the time-0 equilibrium is:

$$P = A_1 D - A_2 Z \quad (6)$$

$$A_1 \equiv I - (\rho^2 \Sigma_{avg}^{-1} \Sigma_Z^{-1} \Sigma_{avg}^{-1} + \Sigma_{avg}^{-1} + \Sigma_D^{-1} + \rho^{-1} k)^{-1} (\rho^{-1} k + \Sigma_D^{-1}) \quad (7)$$

$$A_2 \equiv \rho^{-1} A_1 \Sigma_{avg}^{-1} \quad (8)$$

$$\text{where } \Sigma_{avg}^{-1} \equiv \int_i \Sigma_i^{-1} di \quad (9)$$

As Admati (1985) discusses, this equilibrium is not particularly illuminating since standard intuitions, including the idea that increases in demand or decreases in supply should raise prices, do not hold. Prices of futures maturing on any particular date depend on fundamentals and demand for all other maturities except in knife-edge cases. Interpreting the equilibrium requires being able to build intuition about products of matrix inverses. Likewise, since the information choice problem is concave in precision, is straightforward to write down sufficient conditions characterizing the optimum, but those conditions do not shed light the features of asset returns which determine information choices.

In order to address that issue, a standard approach in the linear rational expectations literature is to express the solution in terms of orthogonal factors – linear combination of fundamentals (portfolios) which span the same space, while remaining mutually independent (e.g. Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016)). In the following section, we show that a natural way to apply that idea to the present model is to form portfolios characterized by the frequencies at which their returns fluctuate.

2 Frequency domain interpretation

2.1 Investment in the frequency domain

Within the model, one can always analyze prices and returns of arbitrary portfolios of the underlying, date-specific futures contracts. In what follows, we construct one specific set of such portfolios, with two key features: they span the same space as the futures contracts; and their returns fluctuate at specific frequencies.

In order to do that, for each frequency $j \in \{0, 1, \dots, T/2\}$, we define two vectors of portfolio weights:

$$c_j \equiv \sqrt{\frac{2}{T}} \left(\cos(\omega_j(t-1)) \right)_{t=1}^T, \quad (10)$$

$$s_j \equiv \sqrt{\frac{2}{T}} \left(\sin(\omega_j(t-1)) \right)_{t=1}^T. \quad (11)$$

c_j is the vector of weights defining the cosine portfolio that fluctuates at frequency $\omega_j \equiv 2\pi j/T$, and s_j is the set of weights defining the sine portfolio that fluctuates at frequency ω_j . A cycle

at frequency ω_j has an associated wavelength $2\pi/\omega_j$. $\omega_0 = 0$ thus corresponds to an infinite wavelength, or a permanent shock (a constant vector). ω_1 corresponds to a cycle that lasts as long as the sample – c_1 has weights that form a single cycle of a cosine. $\omega_{T/2} = \pi$, the highest frequency, corresponds to a cycle that lasts two periods, so that $c_{T/2}$ oscillates between $\pm\sqrt{2/T}$ in each period. Figure 1 plots the weights across maturities, for portfolios of different characteristic frequencies.

In order for these portfolios to fully span the set of futures, a matrix formed from them must have full rank. We denote that matrix of the portfolio weights using Λ and set:

$$\Lambda \equiv \left[\frac{1}{\sqrt{2}}c_0, c_1, \dots, c_{T/2-1}, \frac{1}{\sqrt{2}}c_{T/2}, s_1, s_2, \dots, s_{T/2-1} \right]. \quad (12)$$

The frequency-domain counterpart to the vector of fundamentals, D , is then

$$d = \Lambda' D. \quad (13)$$

From an economic standpoint, d is the vector of fundamentals associated with each of the frequency-specific portfolios. Mathematically, it is simply the discrete Fourier transform of the fundamentals process. We use the notation $d_j = c'_j D$ and $d_{j'} = s'_j D$ to refer to fundamentals of the sine and cosine portfolios associated with frequency j (for $1 \leq j \leq T/2 - 1$ where there is ambiguity). So when the distinction is necessary, the notation j refers to frequencies associated with cosine transforms and j' to sine transforms. In what follows, lower-case letters denote frequency-domain objects.

As a simple example, consider the case with $T = 2$. The low-frequency component of dividends is then $d_0 = (D_1 + D_2)/\sqrt{2}$ and the high-frequency component is $d_1 = (D_1 - D_2)/\sqrt{2}$. Investors trade the low-frequency component d_0 by buying an equal amount of the claims on D_1 and D_2 . Conversely, investors trade the high-frequency component d_1 by buying offsetting amounts of the claims on D_1 and D_2 .⁸

Aside from allowing us to form frequency-specific portfolios, the matrix of portfolio weights Λ has the key property that it approximately diagonalizes all stationary time series.

Lemma 1 *For a stationary time series X_t ,*

$$x \equiv \Lambda' X \Rightarrow N(0, \text{diag}(f_X)) \quad (14)$$

where f_X denotes the spectrum of X_t , and “ \Rightarrow ” denotes convergence in weak matrix norm at rate $T^{-1/2}$.

Proof. This is a textbook result (e.g. Brockwell and Davis (1991)). See appendix A.1 for a derivation specific to our case. ■

⁸One explicit example of this type of trade is the calendar spread future (Cuny (2006)).

Λ diagonalizes the supply process, the price process, the dividend process, and agents' signals. So the frequency portfolios have asymptotically independent dividends and prices. That independence will substantially simplify our analysis. Of course, the matrix Λ does not exactly diagonalize the covariance matrices of D , P , Z , or Y_i at finite horizons. But as T grows, the error induced by ignoring the off-diagonal elements of the covariance matrices of frequency domain objects like $d = \Lambda'D$ become negligible (it is of order $T^{-1/2}$), and d is well approximated as a vector of independent random variables.⁹ The spectrum of D , f_D , measures the variance in D coming from fluctuations at each frequency.¹⁰

2.2 Market equilibrium in the frequency domain

Instead of solving jointly for the prices of all futures, the approximate diagonalization result allows us to solve a series of parallel scalar problems, one for each frequency. Each of these problems should be thought of as an equilibrium in a market where one the frequency-specific portfolio is traded. We obtain a standard and intuitive solution:

Solution 1 *Under the approximations $d \sim N(0, \text{diag}(f_D))$ and $z \sim N(0, \text{diag}(f_Z))$, the prices of the frequency-specific portfolios, p_j , satisfy, for all j, j'*

$$p_j = a_{1,j}d_j - a_{2,j}z_j \quad (15)$$

$$a_{1,j} \equiv 1 - \frac{\rho^{-1}k + f_{D,j}^{-1}}{\left(\rho f_{\text{avg},j}^{-1}\right)^2 f_{Z,j}^{-1} + f_{\text{avg},j}^{-1} + f_{D,j}^{-1} + \rho^{-1}k} \quad (16)$$

$$a_{2,j} \equiv \frac{a_{1,j}}{\rho f_{\text{avg},j}^{-1}} \quad (17)$$

where $f_{\text{avg},j}^{-1} \equiv \int_i f_{i,j}^{-1} di$ is the average precision of the agents' signals at frequency j and f_i is the spectrum of Σ_i . See appendix A.2 for the derivation.

The price of the frequency- j portfolio depends only on fundamentals and supply at that frequency. As usual, the informativeness of prices, through $a_{1,j}$, is increasing in the precision of the signals that investors obtain, while the impact of supply on prices is decreasing in signal precision and risk tolerance. The frequency domain analog to the usual demand function is

$$q_{i,j} = \rho \frac{E[d_j - p_j \mid y_{i,j}, p_j]}{\text{Var}[d_j - p_j \mid y_{i,j}, p_j]}. \quad (18)$$

⁹For all the stationary processes studied in the paper, we assume that the autocovariances are summable in the sense that $\sum_{r=1}^{\infty} |j\sigma_{X,j}|$ is finite (which holds for finite-order stationary ARMA processes, for example).

¹⁰Finally, it is worth noting that infill asymptotics, where T grows by making the length of a time period shorter, are not sufficient for lemma 1 to hold. What is important is essentially that T is large relative to the range of autocorrelation of the process X . So, for example, if fundamentals have nontrivial autocorrelations over a horizon of a year, then it is important that T be substantially larger than a year. Van Binsbergen and Koijen (2017), for example, examine data on dividend futures with maturities as long as 16 years.

These solutions for the prices and demands are the standard results for scalar markets. What is different here is simply that the agents chose exposures across frequencies, rather than across dates; p_j is the price of a portfolio whose exposure to fundamentals fluctuates over time at frequency $2\pi j/T$. Both prices and demands at frequency j depend only on signals and supply at frequency j – the problem is completely separable across frequencies.

Proposition 1 in the appendix shows that the frequency domain solution to the market equilibrium provides a close approximation to the true solution, in the sense that the solution in (15), once it is rotated back to the time domain, converges to equations (6-8). The proposition shows that convergence occurs at rate $T^{-1/2}$; that is, for large T , the standard time-domain solution for stationary time series processes becomes arbitrarily close to a simple set of parallel scalar problems in the frequency domain. The time domain solution is obtained from the frequency domain solution by premultiplying by Λ .

2.3 Optimal information choice in the frequency domain

The analysis above takes the precision of the signals as fixed. Following Van Nieuwerburgh and Veldkamp (2009) and Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016; KVVN) closely, we now allow investors to choose their signal precisions, Σ_i^{-1} .

We assume that investors choose information to maximize the expectation of their mean-variance objective (5) subject to an information cost,

$$\max_{\{f_{i,j}\}} E_{-1} [U_{i,0} | \Sigma_i^{-1}] - \frac{\psi}{2} T^{-1} \text{tr} (\Sigma_i^{-1}), \quad (19)$$

where E_{-1} is the expectation operator on date -1 , i.e. prior to the realization of signals and prices (as distinguished from $E_{i,0}$, which conditions on P and Y_i), and ψ is the per-period cost of information. Total information here is measured by the trace operator $\text{tr} (\Sigma_i^{-1})$. Since the trace of a matrix is equal to the sum of its eigenvalues, this measure of information is the same as summing the total precision of the independent components of the signals. Moreover, since the trace operator is invariant under rotations, our measure of information is invariant to the domain of analysis, time or frequency,

$$\text{tr} (\Sigma_i^{-1}) = \sum_{j,j'} f_{i,j}^{-1}. \quad (20)$$

The information constraint is linear in the frequency-specific precisions. Investors also face the constraint that $f_{i,j} = f_{i,j'}$, which ensures that ε_i is a stationary process.¹¹

The appendix shows that, given the optimal demands, an agent's expected utility is linear in the precision they obtain at each frequency.

¹¹KVVN show that the results here are robust to various perturbations of the assumptions: (1) rather than using the trace operator, information can be measured through the entropy of the signals; (2) investors can be given a fixed budget of information rather than a fixed cost; (3) it can be made costly for investors to pay attention to prices in addition to their signals.

Lemma 2 *When informed investors optimize, each investor's expected utility may be written as a function of their own precisions, $f_{i,j}^{-1}$, and the average across other investors, $f_{avg,j}^{-1} \equiv \int_i f_{i,j}^{-1} di$, with*

$$E_{-1} [U_{0,i} | \{f_{i,j}\}] = \frac{1}{2T} \sum_{j,j'} \lambda_j \left(f_{avg,j}^{-1} \right) f_{i,j}^{-1} + \Gamma, \quad (21)$$

where Γ is a constant that does not depend on investor i 's precision.

$\lambda_j(x)$ is a function determining the marginal benefit of information at each frequency with the properties $\lambda_j(x) > 0$ and $\lambda_j'(x) < 0$ for all $x \geq 0$. The frequency-domain transformation is what allows us to write utility as a simple sum across frequencies; an investor's utility depends additively on the amount of information that they obtain at each frequency. In the time domain, by contrast, the utility of investors would be a more complicated function of the covariance matrices of fundamentals.

Since expected utility and the information cost are both linear in the set of precisions that agent i chooses, $\{f_{i,j}^{-1}\}$, it immediately follows that agents purchase signals at whatever subset of frequencies has $\lambda_j \left(f_{avg,j}^{-1} \right) \geq \psi$.

Solution 2 *Information is allocated so that*

$$f_{avg,j}^{-1} = \begin{cases} \lambda_j^{-1}(\psi) & \text{if } \lambda_j(0) \geq \psi \\ 0 & \text{otherwise} \end{cases}. \quad (22)$$

This is the reverse water-filling solution from KVVV. Since attention cannot be negative, when $\lambda_j(0) \leq \psi$, no attention is allocated to frequency j . Otherwise, attention is allocated so that its marginal benefit and its marginal cost are equated. Note, though, that this result does not pin down precisely how any specific investor's attention is allocated; this class of models, with a non-convex information cost, only determines the aggregate allocation of attention across frequencies. For the purposes of studying price informativeness, though, characterizing this aggregate allocation is all that is necessary.

The allocation result can be interpreted more concretely through the following observation.

Result 1 *The return at frequency j has variance*

$$Var[r_j] = \lambda_j \left(f_{avg,j}^{-1} \right), \quad (23)$$

where $r_j \equiv d_j - p_j$.

The marginal benefit of acquiring information at a particular frequency is exactly equal to the unconditional variance of returns at that frequency. When a specific frequency component of returns has high variance, there are potentially large profits to be earned from acquiring information about it. At frequencies where returns have zero variance, on the other hand, prices are perfectly

informative about dividends. There is no reason to study fundamentals at such a frequency. In equilibrium, agents will therefore choose signals that provide information about frequencies where returns are most volatile.

3 The consequences of restricting investment frequencies

This section focuses on the effects of restricting the frequencies at which investment strategies can operate. Real-world examples of such restrictions abound. Some institutional investors face constraints on the speed at which they can change their portfolio weights. For example, a pension fund or endowment might have a policy portfolio that it targets, the weights of which are only updated on an annual or quarterly basis at board meetings. Other investors have restrictions that keep them from holding positions for too long. Market makers and trading desks may have policies restricting their positions to net to zero at the end of each day (e.g. Brock and Kleidon (1992) and Menkveld (2013)). More directly relevant to the exact model studied in this paper, some investors who trade in futures markets are restricted from having exposure to underlying factors, like a level or slope factor.

Those constraints on portfolio managers are in a sense imposed by their own investors. Regulators may also impose restrictions on the types of strategies that investors may undertake. Some of those policies are aimed at investors who trade at the very highest frequencies (such as the CFTC’s recently proposed Regulation AT; see CFTC (2016)). But there are also proposals to discourage portfolio turnover at the monthly or annual level. The US tax code, for example, encourages holding assets for at least a year through the higher tax rates on short-term capital gains.¹²

The model developed in section 2 has the requisite ingredients to study the potential effects of these restrictions on price informativeness, return volatility, and investor profits. Endogenous information acquisition allows us to speak meaningfully about how the amount of information embedded in price changes in response to the policy. Meanwhile, our results on the underlying frequency structure of the equilibrium make it straightforward to impose frequency-specific restrictions on investment strategies.

3.1 Restricting investment frequencies

We study equilibria in which the sophisticated investors of the model are restricted from following strategies with components that fluctuate at some set of frequencies \mathcal{R} . Specifically, investors are restricted to setting $q_{i,j} = 0$ for $j \in \mathcal{R}$. An investor who is restricted from following strategies with cycles that last less than a quarter, for example, would be required to hold a portfolio of futures that varies smoothly across days, weeks, and months, but that can have reversals over longer horizons. At the most extreme, we could imagine an investor who is allowed to hold a portfolio only at

¹²There have been recent proposals to further expand such policies (a plan to create a schedule of capital gains tax rates that declines over a period of six years was attributed to Hillary Clinton during the 2016 US Presidential election; see Auxier et al. (2016)).

frequency 0. That portfolio would then put equal weight on all futures, thus amounting to a bet on the average level of fundamentals. Restrictions on all frequencies but 0 and ω_1 would allow investors to take positions on not just the average level of fundamentals, but also on the path of fundamentals, and whether they will be stronger in the first or the second half of the sample.

3.2 Effects of the policy

3.2.1 Prices and their informativeness

Result 2 *When trade by sophisticated investors is restricted at a set of frequencies \mathcal{R} , prices satisfy*

$$p_j = \left\{ \begin{array}{l} -k^{-1}z_j \text{ for } j \in \mathcal{R} \\ a_{1,j}d_j - a_{2,j}z_j \text{ otherwise} \end{array} \right\}, \quad (24)$$

where a_1 and a_2 are the same as those defined in solution 1.

The direct effect of the restriction is to cause prices at the restricted frequencies to become completely uninformative about dividends, and instead to only depend on supply. Moreover, the market is completely illiquid in the sense that when exogenous supply increases, there is no change in trade – prices just move so that trade remains at zero. In other words, prices equilibrate the market instead of quantities.

At the frequencies not targeted by the policy, though, there is no effect. The equilibrium for prices conditional on information choices is the same as without the restriction. And since the solution for information acquisition at a frequency j does not depend on anything about any other frequency, the information acquired at a frequency $j \notin \mathcal{R}$ is also unaffected by the policy. We then have the result that:

Result 3 *When investors are restricted from holding portfolios with weights that fluctuate at some set of frequencies $j \in \mathcal{R}$, then prices at those frequencies, p_j , become completely uninformative about dividends. The informativeness of prices for $j \notin \mathcal{R}$ about dividends is unchanged. More formally, $\text{Var}[d_j | p_j]$ for $j \notin \mathcal{R}$ is unaffected by the restriction. For $j \in \mathcal{R}$, $\text{Var}[d_j | p_j] = \text{Var}[d_j]$.*

The fact that prices remain equally informative at some frequencies does not mean that they remain equally informative for any particular *date*. Specifically, the appendix shows that

$$\text{Var}(D_t | P) = \frac{1}{T} \sum_{j,j'} \text{Var}[d_j | p_j]. \quad (25)$$

The variance of an estimate of fundamentals conditional on prices at a particular date is equal to the average of the variances across all frequencies. So when uncertainty rises at some set of frequencies, the informativeness of prices for fundamentals on every date falls by an equal amount.

Result 4 *Investment restrictions reduce price informativeness for fundamentals on all dates by equal amounts, and by an amount that weakly increases with the number of frequencies that are restricted.*

If a person is making decisions based on estimates of fundamentals from prices and they are worried that prices are contaminated by high-frequency noise due to a restriction on high-frequency investment, a natural response would be to examine an average of fundamentals and prices over time (across maturities of futures contracts).

Result 5 *The variance of an estimate of the average of fundamentals over dates t to $t + n - 1$ conditional on observing the vector of prices, P , is*

$$\text{Var} \left(\frac{1}{n} \sum_{m=0}^{n-1} D_{t+m} \mid P \right) = \frac{1}{nT} \sum_{j,j'} F_n(\omega_j) \text{Var} [d_j \mid p_j] \quad (26)$$

$$\text{where } F_n(\omega_j) \equiv \frac{1}{n} \frac{1 - \cos(n\omega_j)}{1 - \cos(\omega_j)} \quad (27)$$

$F_n(\omega_j)$ concentrates its weight on low frequencies as n grows, so if investment is restricted only at high frequencies, price informativeness for long moving averages of fundamentals is unaffected.¹³

$F_n(\omega)$ is the Fejér kernel. $F_1(\omega) = 1$, and as n rises, the mass of the Fejér kernel migrates towards the origin. That is, it places progressively less mass on high frequencies and more on low frequencies. Specifically,

$$\frac{1}{T} \sum_{j,j'} F_n(\omega_j) = 1 \quad (28)$$

$$\lim_{n \rightarrow \infty} F_n(\omega) = 0 \text{ for all } \omega \neq 0 \quad (29)$$

The total weight allocated across the frequencies always sums to 1, and as n rises, the mass becomes allocated eventually purely to frequencies local to zero. Figure 2 illustrates this.

Figure 2 plots F_n for a range of values of n . As n rises, the weight falls toward zero at a progressively wider range of frequencies. Equation (26) therefore shows that while a reduction in precision at high frequencies due to trading restrictions will reduce the informativeness of prices about fundamentals on any single date, it has quantitatively small effects on the informativeness of prices for fundamentals over longer periods.

Result 5 shows that the informativeness of prices for moving averages of fundamentals places relatively more weight on low- than on high-frequency precisions. So even if prices have little or no information at high frequencies – $\text{Var} [d_j \mid p_j]$ is high for large j – there need not be any degradation of information about averages of fundamentals over multiple periods, as they depend primarily on precision at lower frequencies (smaller values of j).

¹³This definition for $F_n(\omega)$ is invalid for $\omega = 0$. More formally, $F_n(\omega) \equiv 1 + 2n^{-1} \sum_{k=0}^{n-1} \sum_{s=1}^k \cos(s\omega)$. These definitions are identical at all other points.

More concretely, going back to our example of oil futures, when investors are not allowed to use high-frequency investment strategies, prices become noisier, making it more difficult to obtain an accurate forecast of the spot price of oil at some specific moment in the future. But the errors in prices have a negatively serially correlated component, so if one is interested in the average of spot oil prices over a year, on the other hand, then we would expect futures prices to remain informative under restrictions on high-frequency strategies. The Fejér kernel formalizes that intuition.

When low-frequency investment strategies are restricted, on the other hand, as in the case of a trading desk that cannot have exposure over cycles lasting longer than a day, then it is natural to examine the informativeness of *differences* in prices across dates. As an example, we can consider the variance of the first difference of fundamentals.

Result 6 *The variance of an estimate of the change in fundamentals across dates conditional on observing the vector of prices is*

$$\text{Var} [D_t - D_{t-1} | P] = \sum_j (2 - 2 \cos(\omega_j)) \text{Var} [d_j | p_j] \quad (30)$$

The function $2 - 2 \cos(\omega)$ is equal to 0 at $\omega = 0$ and rises smoothly to 4 at the highest frequency, $\omega = \pi$. So period-by-period changes in fundamentals are driven primarily by high-frequency variation. Reductions in price informativeness at low frequencies then have relatively large effects on moving averages and small effects on changes, while the reverse is true for reductions in informativeness at high frequencies.

To summarize, any restriction on investment reduces price informativeness for any particular date. But when high-frequency investment is restricted, there is little change in the behavior of moving averages of prices. So if a manager is making investment decisions based on fundamentals only at a particular moment, then that decision will be hindered by the policy since prices now have more noise. But if decisions are made based on averages of fundamentals over longer periods, e.g. over a year, then the model predicts that there need not be adverse consequences.

3.2.2 Return volatility

Result 7 *Given an information policy $f_{avg,j}^{-1}$, the variance of returns at frequency j is*

$$\text{Var} (r_j) = \begin{cases} f_{D,j} + \frac{f_{Z,j}}{k^2} & \text{for } j \in \mathcal{R} \\ \min(\psi, \lambda_j(0)) & \text{otherwise} \end{cases}, \quad (31)$$

Moreover, the variance of returns at restricted frequencies satisfies $\text{Var}(r_j) > f_{D,j} + \frac{f_{Z,j}}{(k + \rho f_{D,j}^{-1})^2}$, which is the variance that returns would have at the same frequency if investment were unrestricted but agents were uninformed.

The volatility of returns at a restricted frequency is higher than it would be if the sophisticated investors were allowed to trade, even if they gathered no information. Intuitively, when uninformed

active investors have risk-bearing capacity ($\rho > 0$), they absorb some of the exogenous supply by simply trading against prices, buying when prices are below their means and selling when they are above. The greater is the risk-bearing capacity, the smaller is the effect of supply volatility on return volatility. Thus, the restriction affects return volatilities through its effects on both liquidity provision and information acquisition.

Restricting sophisticated investors from following high-frequency strategies in this model can thus substantially raise asset return volatility at high frequencies – it can lead to, for example, large minute-to-minute fluctuations in prices (though those fluctuations in prices are, literally, variations in prices across maturities for different futures contracts on date 0). Sophisticated traders typically play a role of smoothing prices across maturities, essentially intermediating between excess inelastic demand in one minute and excess inelastic supply in the next. When they are restricted from holding positions in futures that fluctuate from minute to minute, they can no longer provide that intermediation service, and volatility at high frequencies increases.

3.2.3 Investor outcomes

One possible rationale for regulating high-frequency investment strategies is that investors following such strategies might somehow hurt agents who follow lower-frequency strategies. In the present model, however, the utility and profits of investors who only trade at some frequency $j \notin \mathcal{R}$ are not increased by the restriction on trade at frequencies in \mathcal{R} .

Utility net of information costs We can consider two versions of the regulatory policy. The first would be implemented at the beginning of period -1, prior to information choices being made. In that case, two things happen. First, we obtain the above results about changes in price informativeness. Second, though, there is no effect on net utility earned at the untargeted frequencies. That result again follows from the separability of the model and the utility function (lemma 2).

Specifically, utility net of information costs can be written as

$$E_{-1} [U_{i,0}] - \frac{\psi}{2} T^{-1} \sum_{j,j'} f_{i,j}^{-1} = \frac{1}{2} T^{-1} \sum_{\substack{j: \lambda_j(0) \geq \psi, \\ j \notin \mathcal{R}}} \left[\psi \left((\rho f_{avg,j}^{-1})^2 f_{Z,j}^{-1} + f_{D,j}^{-1} \right) - 1 \right] \quad (32)$$

$$- \frac{\psi}{2} T^{-1} \sum_{j \in \mathcal{R}} f_{i,j}^{-1}$$

The term on the second line represents the loss due to the purchase of information at restricted frequencies. When the policy is announced prior to information choices, investors obviously set $f_{i,j}^{-1} = 0$ for all $j \in \mathcal{R}$. That said, there is still a loss in utility from the restriction, since investors can no longer gather information and trade with the liquidity supply, Z , at the targeted frequencies. In terms of the equation above, that fact is embodied in a narrowing of the domain of the first summation, $j \notin \mathcal{R}$. So there is a direct effect of the policy, which is to reduce the set of trading opportunities that the sophisticated investors have. There are no further effects on utility, though.

The part of utility that comes from any particular frequency $j \notin \mathcal{R}$ is unchanged when the trading restriction is imposed.

An alternative specification with slightly different results is to assume that the restriction on investment strategies is announced at the end of period -1, after information choices have been made, but prior to signals and noise trader demand being revealed. In that case, the investors who purchased information at regulated frequencies lose their investment, since that information no longer has any use. That is, when \mathcal{R} is announced after precisions are chosen, some investors may choose $f_{i,j}^{-1} > 0$ for $j \in \mathcal{R}$, and that (costly) information is now useless. More concretely, if a person pays some sort of fixed cost to become a high-frequency trader and high-frequency trade is restricted, those fixed costs represent a loss.

Profits Beyond utility, another question is how investors' expected profits from trading are affected by the policy.

Result 8 *Investor i 's expected profits (which are also equal to the covariance of their positions with returns) are*

$$E_{-1} [Q'_i R] = \sum_{j \notin \mathcal{R}} \left(E [z_j r_j] + \rho \left(f_{i,j}^{-1} - f_{avg,j}^{-1} \right) Var [d_j - p_j] \right) \quad (33)$$

Expected profits in depend on two terms. The first is profits from liquidity provision. All investors, no matter their information, are able to earn some profits by selling when prices are above zero (their unconditional mean) and buying when prices are below zero, thus providing liquidity to the market.

The second term represents gains or losses depending on how much information the investor has compared with the average sophisticated trader. When an investor is more informed than average, they are able to earn excess profits. Intuitively, investors with $f_{i,j}^{-1} = 0$ are forced to set demand as a function only of prices, while investors with higher precision can also use their signals, which increases the covariance of their holdings with the realization of fundamentals.¹⁴

Since the term involving $\left(f_{i,j}^{-1} - f_{avg,j}^{-1} \right)$ sums to zero across investors, it represents a reallocation of profits among sophisticated investors. Information that the sophisticated investors acquire has no effect on liquidity provision – in any case, they must absorb the supply provided by the noise traders. Information acquisition then has two effects. First, it is a competitive advantage between the sophisticated investors, as reflected in (33). Second, it increases utility by reducing the uncertainty that the investors face.

Since investors can always earn returns identically equal to zero simply by setting $q_{i,j} = 0$ in all states of the world, $E_{-1} [q_{i,j} r_j] \geq 0$. So any increase in the size of the restricted set \mathcal{R} must always (weakly) reduce the total expected profits of the sophisticated investors (since information choices are independent across frequencies). When the set \mathcal{R} is larger, investors have fewer frequencies

¹⁴See Goldstein and Yang (2015) for a detailed analysis of the interaction between trading strategies and information acquisition.

to take advantage of. However, as above, at the unrestricted frequencies there are no effects. A change in profitability or informativeness at one frequency has no effect on any others.

3.3 Robustness

The effects of the policy on the informativeness of prices, on return volatility, and on investors' profits are robust to a number of perturbations to the setup of section 2. Appendix C discusses four types of perturbations: changes to the information acquisition technology; costly learning from prices; non-stationary fundamentals; and changes in time horizon. It shows that results remain largely unchanged. The insight that the model can be orthogonalized across frequencies holds regardless of the particulars of market structure or information costs. So agents' information acquisition decisions across frequencies are mutually independent in those perturbations of the baseline model as well. As a result, the effect of policies are also similar.

3.4 Quadratic trading costs

The restriction that investors have *exactly* zero exposure at certain frequencies is a natural one to study in the model. But there are other ways of imposing limits on investors' exposures across frequencies. We now examine the equilibrium when there are quadratic costs of trading. We argue that, relative to the frictionless benchmark, introducing these costs has analogous effects to the more abstract restriction $q_{i,j} = 0$ for $j \in \mathcal{R}$. Changes in trading costs could be caused either by the imposition of a quadratic tax on shares traded, or by changes in the trading technology.

The model obviously does not have trade over time. However, the exposures that investors choose in the futures market can be replicated through a commitment to trade (at a fixed price) the fundamental on future dates. That is, define a date- t equity claim to be an asset that pays dividends equal to the fundamental on each date from $t+1$ to T . Since the futures contracts involve exchanging money only at maturity, the date- t cost of an equity claim is $P_t^{equity} = \sum_{j=1}^{T-t} R_f^{-j} P_{t+j}$ where R_f is the riskless discount rate, which we assume to be constant. An investor's exposure to fundamentals on date t , $Q_{i,t}$ can be acquired either by buying $Q_{i,t}$ units of forwards on date 0 or by holding $Q_{i,t}^{EQ}$ units of equity entering date t . In the latter case, the volume of trade by investor i would be equal to the change in $Q_{i,t}$ over time. That is, $\Delta Q_{i,t}^{EQ} = \Delta Q_{i,t}$.

We assume that investors now maximize the following objective:

$$U_{0,i} = \max_{\{Q_{i,t}\}} E_{0,i} \left[T^{-1} \sum_{t=1}^T Q_{i,t} (D_t - P_t) \right] - \frac{1}{2} c T^{-1} E_{0,i} \left[\sum_{t=2}^T (Q_{i,t} - Q_{i,t-1})^2 \right] - \frac{1}{2} b T^{-1} E_{0,i} \left[\sum_{t=2}^T Q_{i,t}^2 \right], \quad (34)$$

where $c \geq 0$ captures quadratic costs associated with trading volume, and $b > 0$ is a cost of holding large positions in the assets. The term involving b replaces the aversion to variance in the benchmark setting. That change is made for the sake of tractability, but its economic consequences are minimal (see, e.g., Kasa, Walker, and Whiteman (2013)).

The appendix then shows that the quadratic variation in $Q_{i,t}$ can be written as

$$\sum_{T^{-1}t=2}^T (Q_{i,t} - Q_{i,t-1})^2 \approx (2\pi)^2 T^{-1} \sum_{j,j'} j^2 q_{i,j}^2, \quad (35)$$

where the approximation becomes an equality as $T \rightarrow \infty$. The quadratic variation in the volume induced by an investor depends on their squared exposures at each frequency multiplied by the frequency itself. Intuitively, when $c > 0$, holding exposure to higher frequency fluctuations in fundamentals is more costly because it requires more frequent portfolio rebalancing.

The equilibrium of the model is described in detail in appendix. Here, we highlight key results and explain how they relate to the previous results on restricting trade frequencies.

Result 9 *When $c > 0$, all else equal, investors' equilibrium signal precision is higher at lower frequencies.*

As shown in appendix, with the assumption of fixed quadratic trading costs, the marginal benefit of increasing precision at frequency j is given by:

$$\frac{1}{2}(cj^2 + b)^{-1} \text{Var} [d_j | p_j, y_{i,j}]^2.$$

In particular, it is declining with both the signal precision and the frequency of exposure. Given that the marginal cost of information is the same across frequencies, investors choose higher signal precisions at lower frequencies, all else equal.

The main result regarding the effect of the quadratic trading cost is the following.

Result 10 *A small increase in trading costs, when starting from zero, reduces information acquisition at all frequencies except frequency 0. The effect is larger at higher frequencies. As a corollary, the effect of an increase in trading costs on price informativeness is weaker at longer horizons.*

The first part of this result suggests that if the goal is to reduce high-frequency trade, then a quadratic tax is a more blunt instrument than placing an explicit restriction on trade at the targeted frequencies. A tax on volume affects all investors, regardless of the strategy that they follow. However, the second part of the result suggests that trading costs affect the highest frequencies most strongly. The quadratic cost thus leads, endogenously, to the same changes in information acquisition studied in the main model; namely, the variance of dividends conditional on prices, $\text{Var}(d_j | p_j)$, falls more at higher frequencies. The corollary regarding price informativeness refers to the fact that the variance moving averages of the form:

$$\text{Var} \left(\frac{1}{n} \sum_{m=0}^{n-1} D_{t+m} | P \right)$$

increase less as a result of the increase in trading costs for longer horizons n . In the extreme case of $n = T$, which corresponds to the frequency 0 component of the signals, the increase in trading costs has in fact *no* effect on equilibrium signal precision and thus price informativeness. This can be seen from the expression for the marginal benefit of signal precision above, which is independent of c when $j = 0$.

Thus, overall, the message of the model with quadratic costs is consistent with the previous analysis. Increasing trading costs leads to less informed trading and the effect is tilted toward high frequencies; at lower frequencies, information acquisition decisions are less impacted. As a result, the effect of the increase on the informativeness of prices for fundamentals at long horizons is limited.

4 Conclusion

The aim of this paper is to understand how regulations that restrict the types of strategies that investors may pursue affect price informativeness. We are specifically interested in regulations that affect the speed with which investors may turn over their positions. In order to study that question, we need to have a setting in which investors can make meaningful decisions about investment strategies and in which they have an endogenous information choice. We develop a simple rotation of the standard noisy rational expectations equilibrium that incorporates trade and information acquisition in a futures market.

We show that in the stationary time series context, the natural orthogonalization, which essentially represents the set of principal components, is a frequency transformation. That is, when we examine portfolios of futures whose weights fluctuate across maturities at different frequencies, those portfolios have returns that are mutually (nearly) orthogonal. The paper's first innovation then is to show that there is a simple solution to the model in our setting with futures contracts that involves a series of parallel scalar problems.

The model is then used to examine what happens when investors are told that they may not hold portfolios whose weights vary across maturities at particular frequencies. For example, they might be told that they may not have exposures that vary across maturities above some frequency.

Our key result is that such a policy has precisely zero effect on the informativeness of prices or the profitability of trading at the untargeted frequencies. This result is a natural consequence of the independence of the problem across frequencies. Another important byproduct of this independence is that restrictions on high-frequency investment have a diminishing impact on price informativeness as the forecast horizon increases.

The setting studied in this paper may be viewed as a very simple benchmark case. It has the minimal features necessary to study the types of policies that we are interested in and shows that in the simplest case, restricting investment strategies has only localized effects on price efficiency. While restricting investment, e.g. restricting high-frequency trade, may have ancillary effects on price informativeness, one needs to add more complexity to the model in order to cause them to

appear.

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A Results on the frequency solution

A.1 Proof of lemma 1

Gray (2006) shows that for any circulant matrix (a matrix where row n is equal to row $n - 1$ circularly shifted right by one column, and thus one that is uniquely determined by its top row), the discrete Fourier basis, $u_j = [\exp(i\omega_j t), t = 0, \dots, T - 1]'$ for $j \in \{0, \dots, T - 1\}$, is the set of eigenvectors.

Let Σ be a symmetric Toeplitz matrix with top row $[\sigma_0, \sigma_1, \dots, \sigma_{T-1}]$. Define a vector σ

$$\sigma \equiv [\sigma_0, \sigma_1 + \sigma_{T-1}, \sigma_2 + \sigma_{T-2}, \dots, \sigma_{T-2} + \sigma_2, \sigma_{T-1} + \sigma_1]' \quad (36)$$

Following Rao (2016), we “approximate” Σ by the circulant matrix $\Sigma_{circ} \equiv circ(\sigma)$, where $circ(x)$ refers to the circulant matrix generated by using any vector x as the top row.

Since Σ_{circ} is symmetrical, one may observe that its eigenvalues repeat in the sense that $u_j' \Sigma_{circ} = u_{T-j}' \Sigma_{circ}$ for $0 < j < T$. Since pairs of eigenvectors with matched eigenvalues can be linearly combined to yield alternative eigenvectors, it immediately follows that the matrix Λ from the main text contains a full set of eigenvectors for Σ_{circ} . The associated eigenvalues are

$$f_{\Sigma_{circ}}(\omega_j) = \sigma_0 + 2 \sum_{t=1}^{T-1} \sigma_t \cos(\omega_j t). \quad (37)$$

We can write this relationship more compactly as:

$$\begin{aligned} \Sigma_{circ} \Lambda &= \Lambda f_{\Sigma} \\ \Lambda' \Sigma_{circ} \Lambda &= f_{\Sigma} \end{aligned}$$

where the $T \times T$ diagonal matrix f_{Σ} is given by:

$$f_{\Sigma} = \text{diag} \left(f_{\Sigma}(\omega_0), f_{\Sigma}(\omega_1), \dots, f_{\Sigma} \left(\omega_{\frac{T}{2}} \right), f_{\Sigma}(\omega_1), f_{\Sigma}(\omega_2), \dots, f_{\Sigma} \left(\omega_{\frac{T}{2}-1} \right) \right)'.$$

The approximate diagonalization of the matrix Σ consists in writing:

$$\begin{aligned} \Lambda' \Sigma \Lambda &= f_{\Sigma} + R_{\Sigma} \\ \text{where } R_{\Sigma} &\equiv \Lambda' (\Sigma - \Sigma_{circ}) \Lambda \end{aligned}$$

By direct inspection of the elements of $\Sigma - \Sigma_{circ}$, one may see that the m, n element of R_{Σ} , denoted

$R_\Sigma^{m,n}$ satisfies (defining λ_m to be the m th column of Λ and $\lambda_{m,n}$ to be its m, n element)

$$R_\Sigma^{m,n} \equiv \lambda_m' (\Sigma - \Sigma_{circ}) \lambda_n \quad (38)$$

$$= \sum_{i=1}^T \sum_{j=1}^T \lambda_{m,i} \lambda_{n,j} (\Sigma - \Sigma_{circ})^{m,n} \quad (39)$$

$$\leq \sum_{i=1}^T \sum_{j=1}^T \frac{2}{T} (\Sigma - \Sigma_{circ})^{m,n} \quad (40)$$

$$\leq \frac{4}{T} \sum_{j=1}^{T-1} j |\sigma_j| \quad (41)$$

where $(\Sigma - \Sigma_{circ})^{m,n}$ is the m, n element of $(\Sigma - \Sigma_{circ})$. So R_Σ is bounded elementwise by a term of order T^{-1} . One may show that the weak norm satisfies $|\cdot| \leq \sqrt{T} |\cdot|_{\max}$, where $|\cdot|_{\max}$ denotes the elementwise max norm, which thus yields the result that $|\Lambda \Sigma \Lambda' - \text{diag}(f_\Sigma)| \leq bT^{-1/2}$ for some b .

In the main text, we refine the notation \Rightarrow as follows: $\Lambda X \Rightarrow N(0, \hat{\Sigma}_X)$ if $\Lambda X \sim N(0, \Sigma_X)$ and $|\hat{\Sigma}_X - \Sigma_X| = \bar{O}(T^{-1/2})$. The notation \bar{O} indicates

$$|A - B| = \bar{O}(T^{-1/2}) \iff |A - B| \leq bT^{-1/2} \quad (42)$$

for some constant b and for all T . This is a stronger statement than typical big- O notation in that it holds for all T , as opposed to holding only for some sufficiently large T .

Trigonometric transforms of stationary time series converge in distribution under more general conditions. See Shumway and Stoffer (2011), Brillinger (1981), and Shao and Wu (2007).

A.2 Derivation of solution 1

To save notation, we suppress the j subscripts indicating frequencies in this section when they are not necessary for clarity. So in this section f_D , for example, is a scalar representing the spectral density of fundamentals at some arbitrary frequency.

A.2.1 Statistical inference

We guess that prices take the form

$$p = a_1 d - a_2 z \quad (43)$$

The joint distribution of fundamentals, signals, and prices is then

$$\begin{bmatrix} d \\ y_i \\ p \end{bmatrix} \sim N \left(0, \begin{bmatrix} f_D & f_D & a_1 f_D \\ f_D & f_D + f_i & a_1 f_D \\ a_1 f_D & a_1 f_D & a_1^2 f_D + a_2^2 f_Z \end{bmatrix} \right) \quad (44)$$

The expectation of fundamentals conditional on the signal and price is

$$E[d | y_i, p] = \begin{bmatrix} f_D & a_1 f_D \end{bmatrix} \begin{bmatrix} f_D + f_i & a_1 f_D \\ a_1 f_D & a_1^2 f_D + a_2^2 f_Z \end{bmatrix}^{-1} \begin{bmatrix} y_i \\ p \end{bmatrix} \quad (45)$$

$$= [1, a_1] \begin{bmatrix} 1 + f_i f_D^{-1} & a_1 \\ a_1 & a_1^2 + a_2^2 f_Z f_D^{-1} \end{bmatrix}^{-1} \begin{bmatrix} y_i \\ p \end{bmatrix} \quad (46)$$

and the variance satisfies

$$\tau_i \equiv \text{Var}[d | y_i, p]^{-1} = f_D^{-1} \left(1 - [1 \quad a_1] \begin{bmatrix} 1 + f_i f_D^{-1} & a_1 \\ a_1 & a_1^2 + a_2^2 f_Z f_D^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ a_1 \end{bmatrix} \right)^{-1} \quad (47)$$

$$= \frac{a_1^2}{a_2^2} f_Z^{-1} + f_i^{-1} + f_D^{-1} \quad (48)$$

We use the notation τ to denote a posterior precision, while f^{-1} denotes a prior precision of one of the basic variables of the model. The above then implies that

$$E[d | y_i, p] = \tau_i^{-1} \left(f_i^{-1} y_i + \frac{a_1}{a_2^2} f_Z^{-1} p \right) \quad (49)$$

A.2.2 Demand and equilibrium

The agent's utility function is (where variables without subscripts here indicate vectors),

$$U_i = \max_{\{Q_{i,t}\}} \rho^{-1} E_{0,i} [T^{-1} Q'_i (D - P)] - \frac{1}{2} \rho^{-2} \text{Var}_{0,i} [T^{-1/2} Q'_i (D - P)] \quad (50)$$

$$= \max_{\{Q_{i,t}\}} \rho^{-1} E_{0,i} [T^{-1} q'_i (d - p)] - \frac{1}{2} \rho^{-2} \text{Var}_{0,i} [T^{-1/2} q'_i (d - p)] \quad (51)$$

$$= \max_{\{Q_{i,t}\}} \rho^{-1} T^{-1} \sum_{j,j'} q_{i,j} E_{0,i} [(d_j - p_j)] - \frac{1}{2} \rho^{-2} T^{-1} \sum_{j,j'} q_{i,j}^2 \text{Var}_{0,i} [d_j - p_j] \quad (52)$$

where the last line follows by imposing the asymptotic independence of d across frequencies (we analyze the error induced by that approximation below). The utility function is thus entirely separable across frequencies, with the optimization problem for each $q_{i,j}$ independent from all others.

Taking the first-order condition associated with the last line above for a single frequency, we obtain

$$\begin{aligned} q_i &= \rho \tau_i E[d - p | y_i, p] \\ &= \rho_i \left(f_i^{-1} y_i + \left(\frac{a_1}{a_2^2} f_Z^{-1} - \tau_i \right) p \right) \end{aligned}$$

Summing up all demands and inserting the guess for the price yields

$$z + k(a_1d - a_2z) = \int_i \rho \left(f_i^{-1}y_i + \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i \right) (a_1d - a_2z) \right) di \quad (53)$$

$$= \int_i \rho \left(f_i^{-1}d + \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i \right) (a_1d - a_2z) \right) di \quad (54)$$

Where the second line uses the law of large numbers. Matching coefficients then yields

$$\int_i \rho \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i \right) di = -a_2^{-1}(1 - ka_2) \quad (55)$$

$$\int_i \rho f_i^{-1}a_1^{-1} + \rho \left(\frac{a_1}{a_2^2}f_Z^{-1} - \tau_i \right) di = k \quad (56)$$

and therefore

$$k - \int_i \rho f_i^{-1}a_1^{-1} = -a_2^{-1}(1 - ka_2) \quad (57)$$

$$\int_i \rho f_i^{-1} = \frac{a_1}{a_2} \quad (58)$$

Now define aggregate precision to be

$$f_{avg}^{-1} \equiv \int_i f_i^{-1} di \quad (59)$$

We then have

$$\tau_i = \frac{a_1^2}{a_2^2}f_Z^{-1} + f_i^{-1} + f_D^{-1} \quad (60)$$

$$\tau_{avg} \equiv \int \tau_i di = (\rho f_{avg}^{-1})^2 f_Z^{-1} + f_{avg}^{-1} + f_D^{-1} \quad (61)$$

Inserting the expression for τ_i into (55) yields

$$a_1 = \frac{\tau_{avg} - f_D^{-1}}{\tau_{avg} + \rho^{-1}k} \quad (62)$$

$$a_2 = \frac{a_1}{\rho f_{avg}^{-1}} \quad (63)$$

The expression for a_1 can be written more explicitly as:

$$a_1 = \frac{\tau_{avg} - f_D^{-1}}{\tau_{avg} + \rho^{-1}k} = \frac{\frac{a_1^2}{a_2^2}f_Z^{-1} + f_{avg}^{-1} + f_D^{-1} + \rho^{-1}k - \rho^{-1}k - f_D^{-1}}{\frac{a_1^2}{a_2^2}f_Z^{-1} + f_{avg}^{-1} + f_D^{-1} + \rho^{-1}k} \quad (64)$$

$$= 1 - \frac{\rho^{-1}k + f_D^{-1}}{(\rho f_{avg}^{-1})^2 f_Z^{-1} + f_{avg}^{-1} + \rho^{-1}k + f_D^{-1}} \quad (65)$$

The expression for a_2 is invalid in the case when $f_{avg}^{-1} = 0$. In that case, we have

$$a_2 = \frac{1}{\rho f_D^{-1} + k} \quad (66)$$

A.3 Accuracy of approximate frequency domain solution

Proposition 1 *The difference between solution 1 and the exact Admati (1985) solution is small in the sense that*

$$|A_1 - \Lambda \text{diag}(a_1) \Lambda'| \leq c_1 T^{-1/2} \quad (67)$$

$$|A_2 - \Lambda \text{diag}(a_2) \Lambda'| \leq c_2 T^{-1/2} \quad (68)$$

for constants c_1 and c_2 , where $|\cdot|$ denotes the matrix weak norm. Furthermore, while prices and demands are stochastic, the time- and frequency-domain solutions are related through an even stronger result

$$e_{\max} [\text{Var}(\Lambda p - P)] \leq c_P T^{-1/2} \quad (69)$$

$$e_{\max} [\text{Var}(\Lambda q_i - Q_i)] \leq c_Q T^{-1/2} \quad (70)$$

where the operator $e_{\max}[\cdot]$ denotes the maximum eigenvalue of a matrix (that is, the operator norm), for constants c_P and c_Q .¹⁵

Proof: Standard properties of norms yield the following result. If $|A - B| = \bar{O}(T^{-1/2})$ and $|C - D| = \bar{O}(T^{-1/2})$, then

$$|cA - cB| = \bar{O}(T^{-1/2}) \quad (71)$$

$$|A^{-1} - B^{-1}| = \bar{O}(T^{-1/2}) \quad (72)$$

$$|(A + C) - (B + D)| = \bar{O}(T^{-1/2}) \quad (73)$$

$$|AC - BD| = \bar{O}(T^{-1/2}) \quad (74)$$

In other words, convergence in weak norm carries through under addition, multiplication, and inversion. Since A_1 is a function of Toeplitz matrices using those operations, it follows that $|\Lambda' A_1 \Lambda - \text{diag}(a_1)| = \bar{O}(T^{-1/2})$, and the same holds for A_2 .

For the variance of prices, we define

$$R_1 \equiv A_1 - \Lambda \text{diag}(a_1) \Lambda' \quad (75)$$

$$R_2 \equiv A_2 - \Lambda \text{diag}(a_2) \Lambda' \quad (76)$$

¹⁵The latter result is stronger in the sense that $e_{\max}[x] \leq y \Rightarrow |x| \leq y$.

$$|Var [P - \Lambda p]| \leq |R_1 \Sigma_D R_1'| + |R_2 \Sigma_Z R_2'| \quad (77)$$

$$\leq |R_1 \Sigma_D| |R_1| + |R_2 \Sigma_Z| |R_2| \quad (78)$$

$$\leq \|\Sigma_D\| |R_1|^2 + \|\Sigma_Z\| |R_2|^2 \quad (79)$$

$$\leq K \left(|R_1|^2 + |R_2|^2 \right) \quad (80)$$

The first line follows from the triangle inequality; the second line comes from the sub-multiplicativity of the weak norm; the third line uses the fact that, as indicated by Gray (2006), for any two square matrices G, H , $\|GH\|_2 \leq \|G\| \|H\|$; and the last line follows from the assumption that the eigenvalues of Σ_D and Σ_Z are bounded by some K .

Since the weak norm is invariant under unitary transformations,

$$|R_i| = |\Lambda' R_i \Lambda| = |\Lambda' A_i \Lambda - \text{diag}(a_i)| \quad , \quad i = 1, 2.$$

Therefore,

$$|Var [P - \Lambda P]| \leq K \left(|\Lambda' A_1 \Lambda - \text{diag}(a_1)|^2 + |\Lambda' A_2 \Lambda - \text{diag}(a_2)|^2 \right) \quad (81)$$

$$= \bar{O} \left(\frac{1}{T} \right) \quad (82)$$

Since $\|\cdot\| \leq \sqrt{T} |\cdot|$, $\|Var [P^c - P]\| = \bar{O} (T^{-1/2})$. ■

A.4 Proof of lemma 2

Inserting the optimal value of $q_{i,j}$ into the utility function, we obtain

$$E_{-1} [U_{i,0}] \equiv \frac{1}{2} E \left[T^{-1} \sum_{j,j'} \tau_{i,j} E [d_j - p_j | y_{i,j}, p_j]^2 \right] \quad (83)$$

$U_{i,0}$ is utility conditional on an observed set of signals and prices. $E_{-1} [U_{i,0}]$ is then the expectation taken over the distributions of prices and signals.

$Var [E [d_j - p_j | y_{i,j}, p_j]]$ is the variance of the part of the return on portfolio j explained by $y_{i,j}$ and p_j , while $\tau_{i,j}^{-1}$ is the residual variance. The law of total variance says

$$Var [d_j - p_j] = Var [E [d_j - p_j | y_{i,j}, p_j]] + E [Var [d_j - p_j | y_{i,j}, p_j]] \quad (84)$$

where the second term on the right-hand side is just $\tau_{i,j}^{-1}$ and the first term is $E [E [d_j - p_j | y_{i,j}, p_j]^2]$ since everything has zero mean. The unconditional variance of returns is

$$Var [d_j - p_j] = (1 - a_{1,j})^2 f_{D,j} + \frac{a_{1,j}^2}{\left(\rho f_{avg,j}^{-1} \right)^2} f_{Z,j} \quad (85)$$

So then

$$E_{-1} [U_{i,0}] = \frac{1}{2} T^{-1} \sum_{j,j'} \left[\left((1 - a_{1,j})^2 f_{D,j} + \frac{a_{1,j}^2}{(\rho f_{avg}^{-1})^2} f_{Z,j} \right) \tau_{i,j} - 1 \right] \quad (86)$$

We thus obtain the result that agent i 's expected utility is linear in the precision of the signals that they receive (since $\tau_{i,j}$ is linear in $f_{i,j}^{-1}$; see equation 60). Now define

$$\lambda_j \left(f_{avg,j}^{-1} \right) \equiv (1 - a_{1,j})^2 f_{D,j} + \left(\frac{a_{1,j}}{\rho f_{avg,j}^{-1}} \right)^2 f_{Z,j}. \quad (87)$$

From equations (??)-(62), λ_j can be re-written as:

$$\lambda_j \left(f_{avg,j}^{-1} \right) = \frac{f_{D,j} \left(f_{D,j}^{-1} + \rho^{-1} k \right)^2 + (\rho f_{avg,j}^{-1})^2 f_{Z,j}^{-1} + f_{Z,j} \rho^{-2}}{\left((\rho f_{avg,j}^{-1})^2 f_{Z,j}^{-1} + f_{D,j}^{-1} + \rho^{-1} k + f_{avg,j}^{-1} \right)^2}. \quad (88)$$

Which can be further decomposed as:

$$\begin{aligned} \lambda_j \left(f_{avg,j}^{-1} \right) &= \frac{1}{\left((\rho f_{avg,j}^{-1})^2 f_{Z,j}^{-1} + f_{D,j}^{-1} + \rho^{-1} k + f_{avg,j}^{-1} \right)^2} \\ &+ \frac{f_{Z,j} \frac{f_{avg,j}^{-1}}{\rho}}{\left((\rho f_{avg,j}^{-1})^2 f_{Z,j}^{-1} + f_{D,j}^{-1} + \rho^{-1} k + f_{avg,j}^{-1} \right)^2} \\ &+ \frac{\rho^{-1} k (1 + f_{D,j}^{-1} \rho^{-1} k)}{\left((\rho f_{avg,j}^{-1})^2 f_{Z,j}^{-1} + f_{D,j}^{-1} + \rho^{-1} k + f_{avg,j}^{-1} \right)^2} \end{aligned}$$

Each of these three terms is decreasing in $f_{avg,j}^{-1}$, so that the function $\lambda_j(\cdot)$ is decreasing.

Finally, expected utility can be written as

$$E_{-1} [U_{i,0}] = \frac{1}{2} T^{-1} \sum_{j,j'} \left[\begin{aligned} &\left((1 - a_{1,j})^2 f_{D,j} + \frac{a_{1,j}^2}{(\rho f_{avg}^{-1})^2} f_{Z,j} \right) f_i^{-1} \\ &+ \left((1 - a_{1,j})^2 f_{D,j} + \frac{a_{1,j}^2}{(\rho f_{avg}^{-1})^2} f_{Z,j} \right) \left((\rho f_{avg}^{-1})^2 f_Z^{-1} + f_D^{-1} \right) - 1 \end{aligned} \right] \quad (89)$$

At frequencies where agents obtain information, we have $\lambda_j \left(f_{avg,j}^{-1} \right) = \psi$, when there is no learning, $f_{avg,j}^{-1} = 0$ and since $f_i^{-1} \geq 0$, that means that $f_i^{-1} = 0$ for all i .

$$E_{-1} [U_{i,0}] = \frac{1}{2} T^{-1} \sum_{j: \lambda_j(0) \geq \psi} \left[\psi f_i^{-1} + \psi \left((\rho f_{avg}^{-1})^2 f_Z^{-1} + f_D^{-1} \right) - 1 \right] \quad (90)$$

B Results on trade restrictions

B.1 Results 2, 3 and 7

When there are no active investors and just exogenous supply, we have that $0 = z_j + kp_j$ and so:

$$p_j = -k^{-1}z_j, \quad (91)$$

$$r_j = d_j - k^{-1}z_j. \quad (92)$$

At those frequencies, following the notation of the rest of the model, that $a_{1,j} = 0$ and $a_{2,j} = k^{-1}$. Because of the separability of information choices across frequencies, the coefficients $a_{1,j}$ and $a_{2,j}$ are unchanged at all other frequencies. Moreover, it is clear that $Var(d_j|p_j) = Var(d_j)$ at the restricted frequencies, since prices now only carry information about supply, which is uncorrelated with dividends.

Note that for any $j \in \mathcal{R}$,

$$Var(r_j) = f_{D,j} + \frac{f_{Z,j}}{k^2}. \quad (93)$$

Additionally, if trading at that frequency were not restricted, but the investors endogenously chose not to allocate any attention to the frequency, the return volatility would be:

$$Var_{unrestr.}(r_j) = \lambda_j(0) = f_{D,j} + \frac{f_{Z,j}}{(k + \rho f_{D,j}^{-1})^2} < Var(r_j).$$

B.2 Results 4, 5 and 6

We have

$$D | P \sim N(\bar{D}, \Lambda \text{diag}(\tau_0^{-1}) \Lambda') \quad (94)$$

where τ_0 is a vector of frequency-specific precisions conditional on prices, as of time 0. Given the independence of prices across frequencies, the j -th element of τ_0 is:

$$\tau_{0,j}^{-1} = Var(d_j | p_j).$$

Using this expression, we can compute:

$$Var(D_t) = 1'_t \Lambda \text{diag}(\tau_0^{-1}) \Lambda' 1_t \quad (95)$$

$$= (\Lambda' 1_t)' \text{diag}(\tau_0^{-1}) (\Lambda' 1_t) \quad (96)$$

$$= \sum_{j,j'} \lambda_{t,j}^2 Var(d_j | p_j) \quad (97)$$

$$= \lambda_{t,0}^2 Var(d_0 | p_0) + \lambda_{t,T/2}^2 Var(d_{\frac{T}{2}} | p_{\frac{T}{2}}) + \sum_{j=1}^{T/2-1} (\lambda_{t,j}^2 + \lambda_{t,j'}^2) Var(d_j | p_j) \quad (98)$$

where $\mathbf{1}_t$ is a vector equal to 1 in its t th element and zero elsewhere and $\lambda_{t,j}$ is the j th trigonometric transform evaluated at t , with

$$\lambda_{t,j} = \sqrt{1/T} \cos(2\pi j(t-1)/T) \quad (99)$$

$$\lambda_{t,j'} = \sqrt{1/T} \sin(2\pi j(t-1)/T) \quad (100)$$

$$\lambda_{t,0} = \sqrt{1/T} \quad (101)$$

$$\lambda_{t,T/2} = \sqrt{1/T} \cos(\pi(t-1)) = \sqrt{1/T} (-1)^{t-1} \quad (102)$$

Note that $\lambda_{t,j}^2 + \lambda_{t,j'}^2 = \frac{1}{T}$ and likewise $\lambda_{t,0}^2 = \lambda_{t,T/2}^2 = \frac{1}{T}$. This proves result 4.

More generally, then

$$\text{Var} \left(\frac{1}{s} \sum_{m=0}^{s-1} D_{t+m} \right) = \frac{1}{s^2} \left(\sum_{m=0}^{s-1} \mathbf{1}_{t+m} \right)' \Lambda \text{diag}(\tau_0^{-1}) \Lambda' \left(\sum_{m=0}^{s-1} \mathbf{1}_{t+m} \right) \quad (103)$$

$$= \frac{1}{s^2} \left(\sum_{m=0}^{s-1} \lambda_{t+m,0} \right)^2 \tau_{0,0}^{-1} + \frac{1}{s^2} \left(\sum_{m=0}^{s-1} \lambda_{t+m,T/2} \right)^2 \tau_{0,T/2}^{-1} \quad (104)$$

$$+ \frac{1}{s^2} \sum_{j=1}^{T/2-1} \left[\left(\sum_{m=0}^{s-1} \lambda_{t+m,j} \right)^2 + \left(\sum_{m=0}^{s-1} \lambda_{t+m,j'} \right)^2 \right] \tau_{0,n}^{-1} \quad (105)$$

where $\tau_{0,j}$ is the frequency- j element of τ_0 . For $0 < n < T/2$

$$\left(\sum_{m=0}^{s-1} \lambda_{t+m,j} \right)^2 + \left(\sum_{m=0}^{s-1} \lambda_{t+m,j'} \right)^2 = \sum_{m=0}^{s-1} \sum_{k=0}^{s-1} \frac{2}{T} \begin{bmatrix} \cos(2\pi n(t+m-1)/T) \cos(2\pi n(t+k-1)/T) \\ + \sin(2\pi n(t+m-1)/T) \sin(2\pi n(t+k-1)/T) \end{bmatrix} \quad (106)$$

Now note that

$$2 \cos(x) \cos(y) + 2 \sin(x) \sin(y) = 2 \cos(x-y) \quad (107)$$

So we have

$$\left(\sum_{m=0}^{s-1} \lambda_{t+m,j} \right)^2 + \left(\sum_{m=0}^{s-1} \lambda_{t+m,j'} \right)^2 = \frac{2}{T} \sum_{m=0}^{s-1} \sum_{k=0}^{s-1} \cos\left(\frac{2\pi j}{T}(m-k)\right) \quad (108)$$

$$= 2 \frac{s}{T} \sum_{m=-(s-1)}^{s-1} \frac{s-|m|}{s} \cos\left(\frac{2\pi j}{T}m\right) \quad (109)$$

$$= 2 \frac{s}{T} F_s \left(\frac{2\pi j}{T} \right) \quad (110)$$

$$= \frac{2}{T} \frac{1 - \cos\left(s \frac{2\pi j}{T}\right)}{1 - \cos\left(\frac{2\pi j}{T}\right)} \quad (111)$$

where F_s denotes the s th-order Fejér kernel. Note that when $s = T$, the above immediately reduces to zero, since $\cos(2\pi j) = 0$. That is the desired result, as an average over all dates should be unaffected by fluctuations at any frequency except zero. For $j = 0$,

$$\left(\sum_{m=0}^{s-1} f_{t+m,0} \right)^2 = \left(\sum_{m=0}^{s-1} \sqrt{1/T} \right)^2 \quad (112)$$

$$= \left(s \frac{1}{T^{1/2}} \right)^2 \quad (113)$$

$$= \frac{s}{T} F_s(0), \quad (114)$$

since $F_s(0) = s$ (technically, this holds as a limit: $\lim_{x \rightarrow 0} F_s(x) = s$). For $j = T/2$,

$$\left(\sum_{m=0}^{s-1} f_{t+m,T/2} \right)^2 = \frac{1}{T} \left(\sum_{m=1}^s (-1)^m \right)^2 = \begin{cases} \frac{1}{T} & \text{for odd } s \\ 0 & \text{otherwise} \end{cases} \quad (115)$$

$$= \frac{s}{T} \frac{1}{s} \left(\frac{\sin(s\pi/2)}{\sin(\pi/2)} \right)^2 = \frac{s}{T} F_s(\pi) \quad (116)$$

So we finally have that

$$\text{Var} \left(\frac{1}{s} \sum_{m=0}^{s-1} D_{t+m} \right) = \frac{1}{sT} \sum_{j,j'} F_s(\omega_j) \tau_{0,j}^{-1} \quad (117)$$

Result 6 follows from similar analysis.

B.3 Result 8

Recall that, omitting the j notation:

$$q_i = \rho \left(f_i^{-1} y_i + \left(\frac{a_1}{a_2} f_Z^{-1} - \tau_i \right) p \right)$$

The coefficient on $\tilde{\varepsilon}_i$ is f_i^{-1} . Straightforward but tedious algebra confirms that the coefficient on d is

$$\rho (f_{avg}^{-1} - f_i^{-1}) (a_1 - 1).$$

The coefficient on z is

$$1 + \rho (f_i^{-1} - f_{avg}^{-1}) a_2.$$

We thus have

$$q_i = \rho (f_{avg}^{-1} - f_i^{-1}) (a_1 - 1) d + (1 + \rho (f_i^{-1} - f_{avg}^{-1})) a_2 z + \rho f_i^{-1} \tilde{\varepsilon}_i \quad (118)$$

Now note that

$$r = (1 - a_1) d + d_2 z \quad (119)$$

So then

$$q_i = \rho (f_i^{-1} - f_{avg}^{-1}) r + \rho f_i^{-1} \tilde{\varepsilon}_i + z \quad (120)$$

The result on the covariance then follows trivially from the fact that $E_{-1} [Q'_i R] = \sum_{j,j'} E_{-1} [q_j r_j]$.

C Robustness

C.1 Alternative technologies for information acquisition

Alternatives to the constant marginal cost of information model can be accommodated. We first look at a solution of the model under a fixed information capacity: $tr(\Sigma_i^{-1}) \leq \bar{f}^{-1}$. This is the assumption made in Van Nieuwerburgh and Veldkamp (2010) and KVNV, among others. In that case, the reverse water-filling solution still applies, but the equilibrium marginal benefit of information is endogenous, instead of being pinned down by the exogenous marginal cost ψ . One additional implication of the policy is that agents re-allocate attention toward un-regulated frequencies, in turn increasing price informativeness and lowering return volatility at those frequencies. One can also allow for other measures of information cost of signals than the trace. Consistent with the results of Van Nieuwerburgh and Veldkamp (2010), the properties of the solution remain the same so long as the costs measure is non-convex in precision; we give here an example with an entropy constraint.

C.1.1 Fixed information capacity

In this case, the problem of an individual investor at time -1 is:

$$\max_{\{f_{i,j}\}} E_{-1} [U_{i,0} | \Sigma_i^{-1}] \text{ such that } tr(\Sigma_i^{-1}) \leq \bar{f}^{-1}. \quad (121)$$

The same linear decomposition of utility than in the main model holds. Following the results of KVNV, information is allocated so that

$$f_{avg,j}^{-1} = \begin{cases} \lambda_j^{-1}(\bar{\lambda}) & \text{if } \lambda_j(0) \geq \bar{\lambda} \\ 0 & \text{otherwise} \end{cases} \quad (122)$$

where $\bar{\lambda}$ is obtained as the solution to

$$\sum_{j,j': \lambda_j(0) > \bar{\lambda}} \lambda_j^{-1}(\bar{\lambda}) = \bar{f}^{-1}. \quad (123)$$

The solution for $\bar{\lambda}$ always exists because the functions $\lambda_j(f_{avg,j}^{-1})$ are strictly positive and decreasing. In particular, the solution must satisfy: $\bar{\lambda} < \max_j \lambda_j(0)$. Under the policy restricting investment frequencies, the equilibrium allocation of attention becomes:

$$f_{avg,j}^{-1} = \begin{cases} \lambda_j^{-1}(\bar{\lambda}_{pol}) & \text{if } \lambda_j(0) \geq \bar{\lambda}_{pol} \text{ and } j \notin \mathcal{R} \\ 0 & \text{otherwise} \end{cases} \quad (124)$$

where the equilibrium marginal value of attention is now given by:

$$\sum_{j,j' \in \mathcal{R}: \lambda_j(0) > \bar{\lambda}} \lambda_j^{-1}(\bar{\lambda}_{pol}) = \bar{f}^{-1}. \quad (125)$$

For any frequency j that remains active across equilibria (of which there must be at least 1), we have $\lambda_j^{-1}(\bar{\lambda}_{pol}) < \lambda_j^{-1}(\bar{\lambda})$; since the functions λ_j are decreasing, this implies that $\bar{\lambda}_{pol} < \bar{\lambda}$. This in turn implies both that attention received by frequencies that remain active is higher, and that return volatility is lower (and price precision higher) at those frequencies.

C.1.2 Entropy-based cost of information

We follow Sims (2003) and KVVN in modeling the attention constraint as a limit in the reduction in the entropy of private signals that investors can achieve by conditioning on new information. Specifically, we assume that investor i faces the constraint:

$$\Delta H_i = H(D) - H(\hat{D}_i) \leq \bar{H},$$

where $H(D)$ denotes the unconditional entropy vector of dividend realizations, $H(\hat{D}_i)$ denotes its entropy conditional on a particular choice of signal precisions, summarized by the variance-covariance matrix Σ_i , and \bar{H} is a scalar.

Let $\Sigma_{D,i}^{-1}$ denote the posterior precision of dividends, and let $\{f_{D,i,j}\}_j$ denote the eigenvalues of this matrix. Using the properties of normal random variables, the change in entropy ΔH_i can be rewritten as:

$$\Delta H_i = \frac{1}{2} \ln \left(\frac{|\hat{\Sigma}_{D,i}^{-1}|}{|\Sigma_{D,i}^{-1}|} \right) = \frac{1}{2} \ln \left(\frac{\prod_j f_{D,i,j}^{-1}}{\prod_j f_{D,j}^{-1}} \right).$$

The constraint is therefore equivalent to:

$$\prod_j f_{D,i,j}^{-1} \leq \bar{f}^{-1}, \quad \bar{f}^{-1} \equiv \exp(2H) \prod_j f_{D,j}^{-1}.$$

Following the same steps as in the linear constraint case for the objective, we can therefore write the investors' problem as:

$$\max_{\{f_{i,j}^{-1}\}} \frac{1}{2T} \sum_{j,j'} \lambda_j \left(f_{avg,j}^{-1} \right) f_{i,j}^{-1} + c \quad \text{such that} \quad \prod_j f_{D,i,j}^{-1} \leq \bar{f}^{-1}, \quad f_{i,j}^{-1} \geq 0,$$

where c is a constant. The posterior precisions of dividends, $\{f_{D,i,j}^{-1}\}_j$, is connected to the prior precisions of the signals, through:

$$f_{D,i,j}^{-1} = f_{D,j}^{-1} + f_{i,j}^{-1}.$$

Note that the equality here, which is obtained by pre- and post-multiplying the equality $\Sigma_{D,i}^{-1} = \Sigma_D^{-1} + \Sigma_i^{-1}$ by the matrix Λ defined in the main text, is approximate, in the sense of equation (15) in lemma (1). The constraint $f_{i,j}^{-1} \geq 0$ is then equivalent to the no-forgetting constraint $f_{D,i,j}^{-1} \geq f_{D,j}^{-1}$. Using this result, we can re-write the attention allocation problem as:

$$\max_{\{f_{D,i,j}^{-1}\}} \frac{1}{2T} \sum_{j,j'} \lambda_j \left(f_{avg,j}^{-1} \right) f_{D,i,j}^{-1} + c' \quad \text{such that} \quad \prod_j f_{D,i,j}^{-1} \leq \bar{f}^{-1}, \quad f_{D,i,j}^{-1} \geq f_{D,j}^{-1},$$

where $c' = c - \frac{1}{2T} \sum_{j,j'} \lambda_j f_{D,j}^{-1}$.

The objective of the problem is unchanged. The constraint is now concave; therefore, the Lagrangian associated with the problem remains convex. The solution to the individual investors' attention allocation problem is the same as that of the linear cost function case. Namely, individual investors allocate all attention to frequencies $j \in \mathcal{J}$, where $\mathcal{J} = \arg \max_j \lambda_j$. All other results, including the water-filling solution, follow.

C.2 Costly learning from prices

An assumption implicit in the model of section 2 is that agents can costlessly learn from prices, whereas acquiring and processing signals about fundamentals is costly. In what follows we show that, when conditioning on prices adds to the information costs borne by agents, in equilibrium, agents chose not to learn from prices. This is a well-know result in this class of models; see, for example, the appendix to KVVN. Relative to the model of section 2, equilibrium prices at all frequencies may then become less informative about fundamentals, in the sense of a higher conditional variance of dividends. However, it is still the case that restricting investment at certain frequencies – *within* the model where agents do not learn from prices – has little effect on the informativeness of prices about other frequencies. Agents' attention allocation decisions remain unchanged at the unregulated frequencies, and so results 3 – 7 follow.

Assume that learning from prices is costly. At that at time -1 , if agent i decides to infer information from prices, then their total cost of information is:

$$\psi \text{Tr}(f_i^{-1} + f_P^{-1}),$$

where f_P^{-1} is inverse of the variance-covariance matrix of signals contained in prices, and f_i^{-1} is the variance-covariance of the private signals of agent i . On the other hand, if agent i decides not to infer information from prices, then their total cost of information is:

$$\psi \text{Tr}(f_i^{-1}).$$

Then, the well-know result is that agents prefer not learn from prices. If agent i has decided not to learn from prices, then at time 0, their posterior distribution over d is:

$$\begin{aligned} d|y_i &\sim N(\mu(y_i), \tau_i^{-1}) \\ \tau_i^{NP} &= f_D^{-1} + f_i^{-1} \\ \mu(y_i) &= (\tau_i^{NP})^{-1} f_i^{-1} y_i \end{aligned} \tag{126}$$

Agent i still observes prices; their first-order condition leads to the demand schedule:

$$q_i = \rho \tau_i^{NP} (\mu(y_i) - p).$$

Their time-0 utility is:

$$U_{0,i}^{NP}(y_i; p) = \frac{1}{2T} (\mu(y_i) - p)' \tau_i^{NP} (\mu(y_i) - p). \tag{127}$$

Since τ_i^{NP} is symmetric, this implies:

$$E_{-1,i} [U_{0,i}^{NP}] = \frac{1}{2T} \text{tr}(\tau_i^{NP} V_i^{NP}) + \frac{1}{2T} (\mu_i^{NP})' \tau_i^{NP} \mu_i^{NP}, \tag{128}$$

where as before:

$$\begin{aligned} \mu_i^{NP} &= E_{-1,i} [\mu(y_i) - p] \\ V_i^{NP} &= \text{Var}_{-1,i} [\mu(y_i) - p] \end{aligned} \tag{129}$$

As before, because all fundamentals are mean 0, $\mu_i = 0$. Moreover, by the law of total variance:

$$V_i = \underbrace{\text{Var}_{-1} [d - p]}_{\equiv V_{-1}} - (\tau_i^{NP})^{-1}$$

Therefore,

$$\begin{aligned} E_{-1,i} [U_{0,i}^{NP}] &= \frac{1}{2T} \text{tr}(\tau_i^{NP} V_i) \\ &= \frac{1}{2T} \text{tr}(\tau_i^{NP} V_{-1}) - \frac{1}{2T} \text{tr}(I) \\ &= \frac{1}{2T} \text{tr}(f_D^{-1} V_{-1}) - \frac{1}{2T} \text{tr}(I) + \frac{1}{2T} \text{tr}(f_i^{-1} V_{-1}) \end{aligned} \tag{130}$$

The time-(-1) attention allocation problem of such an agent is therefore:

$$\begin{aligned} U_{-1,i}^{NP}(f_{avg}^{-1}) &= -\frac{1}{2} + \frac{1}{2T} \text{tr}(f_D^{-1} V_{-1}) + \frac{1}{2T} \max_{f_i^{-1}} \text{tr}(f_i^{-1} V_{-1}) - \psi \text{tr}(f_i^{-1}) \\ \text{s.t.} \quad &f_{i,j}^{-1} \geq 0 \quad \forall j \in [0, \dots, T-1] \end{aligned} \tag{131}$$

For an agent who does learn from prices (but shares the other agent's ex-ante distribution over p and d , summarized by V_{-1}), the attention allocation problem has already been derived; it is given

by:

$$\begin{aligned}
U_{-1,i}(f_{avg}^{-1}) &= -\frac{1}{2} + \frac{1}{2T}tr((f_D^{-1} + f_P^{-1})V_{-1}) + \frac{1}{2T} \max_{f_i^{-1}} tr(f_i^{-1}V_{-1}) - \psi tr(f_i^{-1} + f_P^{-1}) \\
\text{s.t.} \quad f_{i,j}^{-1} &\geq 0 \quad \forall j \in [0, \dots, T-1]
\end{aligned} \tag{132}$$

For any diagonal matrix X , $X \rightarrow tr(XV_{-1})$ can be thought of as a linear map on R^T . By the Riesz representation theorem, there is $\mu \in R^T$ such that $\forall X \in diag(R^T)$, $tr(XV_{-1}) = \sum_{j=0}^{T-1} X_{i,j}^{-1} \mu_j$. Let $\tilde{\mu}$ denote the element-wise maximum of μ . In any equilibrium, it must be the case that $\max_j \mu_j \leq \psi$ and, for any frequency receiving a positive amount of attention, $\mu_j = \psi$. After optimization, not learning through prices therefore yields utility:

$$U_{-1,i}^{NP}(f_{avg}^{-1}) = -\frac{1}{2} + \frac{1}{2T}tr(f_D^{-1}V_{-1}),$$

whereas learning through prices yields utility:

$$U_{-1,i}(f_{avg}^{-1}) = -\frac{1}{2} + \frac{1}{2T}tr((f_D^{-1} + f_P^{-1})V_{-1} - \psi f_P^{-1})$$

The difference between the two is:

$$\begin{aligned}
U_{-1,i}^{NP}(f_{avg}^{-1}) - U_{-1,i}(f_{avg}^{-1}) &= \frac{1}{2T} \sum_j f_{P,j}^{-1} (\psi - \mu_j) \\
&\geq 0
\end{aligned} \tag{133}$$

Therefore, the agent always prefer not to learn from prices. Omitting the j, j' notation for clarity, the solution for equilibrium prices in this case is:

$$p = a_3 d - a_4 z$$

with a_3, a_4 diagonal matrices of size $T \times T$. Straightforward derivations lead to:

$$\begin{aligned}
a_3 &= I - (\tau_{avg} + kI)^{-1} (f_D^{-1} + kI) \\
&= (\tau_{avg} + kI)^{-1} f_{avg}^{-1} \\
a_4 &= \frac{1}{\rho} a_3 f_{avg} \\
&= \frac{1}{\rho} (\tau_{avg} + kI)^{-1} \\
\tau_{avg} &= f_{avg}^{-1} + f_D^{-1} \\
\tau_i &= f_i^{-1} + f_D^{-1}
\end{aligned} \tag{134}$$

The variance of dividends conditional on prices will *relative to* the baseline model where agents

learn from prices; in particular, one can show that it is strictly larger so long as $\rho > 1$. However, *within* the model where agents do not learn through prices, results 3-7 remain true, replacing $a_{1,j}$ by $a_{3,j}$ and $a_{2,j}$ by $a_{4,j}$.

C.3 Stationarity and time horizon

The assumption of (trend) stationarity of the underlying fundamentals processes might seem restrictive. The analysis is however similar if a transformation of D_t (e.g. its first difference) has the required properties, as discussed below. Additionally, it could seem that a higher time horizon T might push investors toward lower-frequency strategy. This intuition is in fact incorrect; T instead influences how precisely strategies can load on specific frequencies.

C.3.1 Non-stationary fundamentals

If fundamentals are non-stationary, e.g. if D_t has a unit root, then Σ_D is no longer Toeplitz and our results do not hold. In that case, we assume that D_0 is known by all agents and that the distribution of $\Delta D_t \equiv D_t - D_{t-1}$ is known, with covariance matrix $\Sigma_{\Delta D}$. Then the entire problem can simply be rescaled by defining $\tilde{P}_t \equiv P_t - D_{t-1}$, so that

$$R_t = D_t - P_t \tag{135}$$

$$= \Delta D_t - \tilde{P}_t \tag{136}$$

Our analysis then applies to \tilde{P}_t and ΔD_t , with $Q_{i,t}$ continuing to represent the number of forward contracts on D_t that agent i buys. That is, we are allowing agents to condition demand $Q_{i,t}$ not just on signals and prices, but also the level of D_{t-1} , simply through differencing.

C.3.2 Time horizon and investment

At first glance, the assumption of mean-variance utility over cumulative returns over a long period of time ($T \rightarrow \infty$) may appear to give investors an incentive to primarily worry about long-horizon performance, whereas a small value of T would make investors more concerned about short-term performance. In the present setting, that intuition is not correct – the $T \rightarrow \infty$ limit determines how detailed investment strategies may be, rather than incentivizing certain types of strategies.

The easiest way to see why the time horizon controls only the detail of the investment strategies is to consider settings in which T is a power of 2. If $T = 2^k$, then the set of fundamental frequencies is

$$\left\{ 2\pi j / 2^k \right\}_{j=0}^{2^{k-1}} \tag{137}$$

For $T = 2^{k-1}$, the set of frequencies is

$$\left\{ 2\pi j / 2^{k-1} \right\}_{j=0}^{2^{k-2}} = \left\{ 2\pi (2j) / 2^k \right\}_{j=0}^{2^{k-2}} \tag{138}$$

That is, when T falls from 2^k to 2^{k-1} , the effect is to simply eliminate alternate frequencies. Changing T does not change the lowest or highest available frequencies (which are always 0 and π , respectively). It just discretizes the $[0, \pi]$ interval more coarsely; or, equivalently, it means that the matrix Λ is constructed from a smaller set of basis vectors.

When T is smaller – there are fewer available basis functions – Q and its frequency domain analog $q \equiv \Lambda'Q$ have fewer degrees of freedom and hence must be less detailed. So the effect of a small value of T is to make it more difficult for an investor to *isolate* particularly high- or low-frequency fluctuations in fundamentals (or any other narrow frequency range). But in no way does T cause the investor's portfolio to depend more on one set of frequencies than another. While we take $T \rightarrow \infty$, we will see that the model's separating equilibrium features investors who trade at both arbitrarily low and high frequencies, and T has no effect on the distribution of investors across frequencies.

D Quadratic costs

D.1 Frequency domain expressions for trading costs

Each agent's position can be written as

$$Q_{i,t} = \sum_j \begin{bmatrix} q_j \cos(2\pi jt/T) \\ +q_{j'} \sin(2\pi jt/T) \end{bmatrix} \quad (139)$$

Trading costs are then written in terms of $(Q_{i,t} - Q_{i,t-1})^2$ as:

$$QV\{Q_i\} \equiv \sum_{t=2}^T (Q_{i,t} - Q_{i,t-1})^2 \approx \sum_{t=2}^T \left(\sum_j \frac{2\pi j}{T} \begin{bmatrix} q_j \sin(2\pi jt/T) \\ +q_{j'} \cos(2\pi jt/T) \end{bmatrix} \right)^2 \quad (140)$$

where the approximation assumes that $\cos(2\pi jt/T) - \cos(2\pi j(t-1)/T) \approx \frac{2\pi j}{T} \sin(2\pi jt/T)$ and the same for the differences in the sines. To calculate the first-order condition for q_k , where k is a particular frequency, we need the following

$$\frac{d}{dq_k} QV\{Q_i\} = \frac{d}{dq_k} \sum_{t=2}^T \left(\sum_j \frac{2\pi j}{T} \begin{bmatrix} q_j \sin(2\pi jt/T) \\ +q_{j'} \cos(2\pi jt/T) \end{bmatrix} \right)^2 \quad (141)$$

$$= \sum_{t=2}^T 2 \left(\sum_j \frac{2\pi j}{T} \begin{bmatrix} q_j \sin(2\pi jt/T) \\ +q_{j'} \cos(2\pi jt/T) \end{bmatrix} \right) \left(\frac{2\pi k}{T} \sin(2\pi kt/T) \right) \quad (142)$$

$$\approx 2q_k \sum_{t=2}^T \left(\frac{2\pi k}{T} \right)^2 \sin(2\pi kt/T)^2 \quad (143)$$

$$\approx q_k (2\pi k)^2 T^{-1} \quad (144)$$

where the third line uses the fact that sines of unequal frequencies are orthogonal (it is approximate because $t = 1$ is not included in the sum) and the fourth line inserts the integral for \sin^2 , rather than the exact finite sum. All the approximations here are accurate for large T . Furthermore, note that the derivatives are independent across frequencies, meaning that we can again treat the problem as separable. That result follows from the orthogonality of $\sin(2\pi j/T)$ and $\sin(2\pi k/T)$ for $j \neq k$. Furthermore, note that the effect of trading costs differs across frequencies. As one might expect, higher frequencies have higher trading costs. The expression for the total holding costs follows from:

$$\sum_t Q_t^2 = \sum_{j,j'} q_j^2 \quad (145)$$

which is just Parseval's theorem.

D.2 Equilibrium of the trading cost model

Throughout the analysis, unless it is necessary, we omit the index j of the particular frequency in order to simplify notation.

D.2.1 Investment and equilibrium

The first-order condition for frequency j is

$$0 = E[d_j - p_j \mid y_{i,j}, p_j] - cj^2q_j - bq_j^2 \quad (146)$$

$$q = \frac{E[d_j - p_j \mid y_{i,j}, p_j]}{cj^2 + b} \quad (147)$$

$$= (cj^2 + b)^{-1} \tau_i^{-1} \left(f_i^{-1} y_i + \left(\frac{a_1}{a_2} f_Z^{-1} - \tau_i \right) p \right) \quad (148)$$

We can then solve for the coefficients a_1 and a_2 as before.

Inserting the formula for the conditional expectation and integrating across investors yields

$$\int_i (cj^2 + b)^{-1} \tau_i^{-1} \left(f_i^{-1} y_i + \left(\frac{a_1}{a_2} f_Z^{-1} - \tau_i \right) (a_1 d - a_2 z) \right) di = z_j \quad (149)$$

$$\int_i (cj^2 + b)^{-1} \tau_i^{-1} \left(f_i^{-1} d + \left(\frac{a_1}{a_2} f_Z^{-1} - \tau_i \right) (a_1 d - a_2 z) \right) di = z_j \quad (150)$$

Matching coefficients then yields

$$\int_i (cj^2 + b)^{-1} \tau_i^{-1} \left(\frac{a_1}{a_2} f_Z^{-1} - \tau_i \right) di = -a_2^{-1} \quad (151)$$

$$\int_i (cj^2 + b)^{-1} \tau_i^{-1} \left(f_i^{-1} + \left(\frac{a_1}{a_2} f_Z^{-1} - \tau_i \right) a_1 \right) di = 0 \quad (152)$$

Combining those two equations, we obtain

$$\int_i (cj^2 + b)^{-1} \tau_i^{-1} f_i^{-1} di = \frac{a_1}{a_2} \quad (153)$$

Now put the definition of τ_i into that equation for f_i^{-1}

$$\int_i (cj^2 + b)^{-1} \tau_i^{-1} \left(\tau_i - \frac{a_1^2}{a_2^2} f_Z^{-1} - f_D^{-1} \right) di = \frac{a_1}{a_2} \quad (154)$$

$$(cj^2 + b)^{-1} \int_i 1 - \left(\frac{a_1^2}{a_2^2} f_Z^{-1} - f_D^{-1} \right) \tau_i^{-1} di = \frac{a_1}{a_2} \quad (155)$$

D.2.2 Expected utility

$$U_{i,j} = q_{i,j} E_{0,i} [d_j - p_j] - \frac{1}{2} q_{i,j}^2 cj^2 - \frac{1}{2} b q_{i,j}^2 \quad (156)$$

$$= \frac{1}{2} \frac{E [d_j - p_j | y_{i,j}, p_j]^2}{cj^2 + b} \quad (157)$$

Expected utility prior to observing signals is then

$$EU_{i,j} \equiv \frac{1}{2} E \left[\frac{E [d_j - p_j | y_{i,j}, p_j]^2}{cj^2 + b} \right] \quad (158)$$

$E [E [d_j - p_j | y_{i,j}, p_j]^2]$ is the variance of the part of the return on portfolio j explained by $y_{i,j}$ and p_j , while $\tau_{i,j}$ is the residual variance. We know from the law of total variance that

$$\text{Var} [d_j - p_j] = \text{Var} [E [d_j - p_j | y_{i,j}, p_j]] + E [\text{Var} [d_j - p_j | y_{i,j}, p_j]] \quad (159)$$

where the second term on the right-hand side is just $\tau_{i,j}^{-1}$ and the first term is $E [E [d_j - p_j | y_{i,j}, p_j]^2]$ since everything has zero mean. The unconditional variance of returns is simply

$$\text{Var} [d_j - p_j] = \text{Var} [(1 - a_1) d_j + a_2 z_j] \quad (160)$$

$$= (1 - a_1)^2 f_{D,j} + a_2^2 f_{Z,j} \quad (161)$$

So then

$$EU_{i,j} = \frac{1}{2} \frac{\text{Var} [d_j - p_j] - \tau_{i,j}^{-1}}{cj^2 + b} \quad (162)$$

What we end up with is that utility is decreasing in $\tau_{i,j}^{-1}$. That is,

$$EU_{i,j} = -\frac{1}{2} \frac{\tau_{i,j}^{-1}}{cj^2 + b} + \text{constants} \quad (163)$$

D.2.3 Information choice

With the linear cost on precision, agents maximize

$$-\frac{1}{2} \frac{\tau_{i,j}^{-1}}{cj^2 + b} - \psi f_{i,j}^{-1} \quad (164)$$

$$= -\frac{1}{2} \left(\frac{a_1^2}{a_2^2} f_{Z,j}^{-1} + f_{i,j}^{-1} + f_{D,j}^{-1} \right)^{-1} (cj^2 + b)^{-1} - \psi f_{i,j}^{-1} \quad (165)$$

The FOC for $f_{i,j}^{-1}$ is

$$\psi = \frac{1}{2} \tau_{i,j}^{-2} (cj^2 + b)^{-1} \quad (166)$$

$$\tau_{i,j} = \frac{1}{\sqrt{2}} \psi^{-1/2} (cj^2 + b)^{-1/2} \quad (167)$$

But τ has a lower bound of $\frac{a_1^2}{a_2^2} f_Z^{-1} + f_D^{-1}$, so it's possible that this has no solution. That would be a state where agents do no learning. Formally,

$$\tau_{i,j} = \max \left(\frac{a_1^2}{a_2^2} f_Z^{-1} + f_D^{-1}, \frac{1}{\sqrt{2}} \psi^{-1/2} (cj^2 + b)^{-1/2} \right) \quad (168)$$

Note that, unlike in the other model, the equilibrium is unique here – all agents individually face a concave problem with an interior solution.

Frequencies with no learning Now using the result for a_1/a_2 from above, at the frequencies where nobody learns, $f_i^{-1} = 0$, we have

$$\frac{a_1}{a_2} = \int_i (cj^2 + b)^{-1} \tau_i^{-1} f_i^{-1} di \quad (169)$$

$$= 0 \quad (170)$$

which then implies

$$\tau_{i,j} = \max \left(f_D^{-1}, \frac{1}{\sqrt{2}} \psi^{-1/2} (cj^2 + b)^{-1/2} \right) \quad (171)$$

To get a_2 , we have

$$\int_i (cj^2 + b) \tau_i^{-1} \left(\frac{a_1}{a_2^2} f_Z^{-1} - \tau_i \right) di = -a_2^{-1} \quad (172)$$

$$cj^2 + b = a_2 \quad (173)$$

So the sensitivity of the price to supply shocks is increasing in the cost of holding inventory, b , and the trading costs, c . It is also higher at higher frequencies – it is harder to temporarily push through supply than to do it persistently.

Frequencies with learning At the frequencies at which there is learning, where

$$f_D^{-1} < \frac{1}{\sqrt{2}} \psi^{-1/2} (cj^2 + b)^{-1/2} \quad (174)$$

we have, just by rewriting the τ equation,

$$f_i^{-1} = \tau_i - \frac{a_1^2}{a_2} f_Z^{-1} - f_D^{-1} \quad (175)$$

Using the second equation from above,

$$\int_i (cj^2 + b)^{-1} \tau_i^{-1} \left(\frac{a_1}{a_2} f_Z^{-1} - \tau_i \right) di = -a_2^{-1} \quad (176)$$

$$\int_i (cj^2 + b)^{-1} \tau_i^{-1} \left(\frac{a_1}{a_2} f_Z^{-1} - a_2 \tau_i \right) di = -1 \quad (177)$$

$$\int_i (cj^2 + b)^{-1} \left(\tau_i^{-1} \frac{a_1}{a_2} f_Z^{-1} - a_2 \right) di = -1 \quad (178)$$

Under the assumption of a symmetric strategy, this is

$$\begin{aligned} \tau^{-1} \frac{a_1}{a_2} f_Z^{-1} - a_2 &= -cj^2 - b \\ \frac{a_1}{a_2} &= \tau f_Z (-cj^2 - b + a_2) \end{aligned}$$

Using the other equilibrium condition, we have

$$\int_i (cj^2 + b)^{-1} \tau_i^{-1} \left(\tau_i - \frac{a_1^2}{a_2} f_Z^{-1} - f_D^{-1} \right) di = \frac{a_1}{a_2} \quad (179)$$

$$\int_i (cj^2 + b)^{-1} \left(1 - \tau_i^{-1} \frac{a_1}{a_2} f_Z^{-1} \frac{a_1}{a_2} - \tau_i^{-1} f_D^{-1} \right) di = \frac{a_1}{a_2} \quad (180)$$

$$1 - (-cj^2 - b + a_2) \frac{a_1}{a_2} - \tau_i^{-1} f_D^{-1} = (cj^2 + b) \frac{a_1}{a_2} \quad (181)$$

$$1 - \tau_i^{-1} f_D^{-1} = a_1 \quad (182)$$

Plugging in the formula for τ_i when there is learning,

$$1 - \sqrt{2} \psi^{1/2} (cj^2 + b)^{1/2} f_D^{-1} = a_1. \quad (183)$$

The expression for a_2 can be obtained from:

$$\frac{a_1}{\tau f_Z} = (-cj^2 - b + a_2) a_2. \quad (184)$$

Since $a_1/\tau f_Z > 0$, we know that there is only one solution to this equation for $a_2 > 0$. The positive root is

$$a_2 = \frac{cj^2 + b + \sqrt{(cj^2 + b)^2 + 4\frac{a_1}{\tau f_Z}}}{2} \quad (185)$$

D.3 Results 9 and 10

Result 9 follows from the fact that, in equilibrium,

$$\tau_{i,j} = \max\left(f_{D,j}^{-1}, \frac{1}{\sqrt{2}}\psi^{-1}(cj^2 + b)^{-\frac{1}{2}}\right).$$

This expression is increasing strictly in j^2 (when $c > 0$), on the range of frequencies j where there is learning, that is:

$$f_{D,j}^{-1} \leq \frac{1}{\sqrt{2}}\psi^{-1}(cj^2 + b)^{-\frac{1}{2}}.$$

The first part of result 10 also follows from this expression, since $\tau_{i,j}$ is strictly increasing in c on the range of active frequencies such that $j > 0$.

For the second part of result 10, differentiate $\tau_{i,j}$ with respect to c and j :

$$\frac{d}{dc}\tau_{i,j} = \frac{1}{2\sqrt{2}\psi} \frac{-j^2}{(cj^2 + b)^{3/2}} \quad (186)$$

$$\frac{d}{dj^2} \left[\frac{d}{dc}\tau_{i,j} \right] = -\frac{1}{2\sqrt{2}\psi} \frac{b - \frac{1}{2}cj^2}{(cj^2 + b)^{5/2}} \quad (187)$$

$$\frac{d}{dj^2} \left[\frac{d}{dc}\tau_{i,j} \right]_{c=0} = -\frac{1}{2\sqrt{2}\psi} b^{-3/2} \quad (188)$$

So for small c , high frequencies are most sensitive. Dividing by total information at $c = 0$ does not change the sign of that derivative. The result on price informativeness requires establishing that:

$$\frac{\partial^2}{\partial c \partial j} [Var(d_j|p_j)]_{c=0} > 0.$$

This result can be obtain using the fact that:

$$Var(d_j|p_j) = \left(f_{D,j}^{-1} + f_{Z,j}^{-1} \left(\frac{a_{1,j}}{a_{2,j}} \right)^2 \right)^{-1},$$

where the ratio $x_j = \frac{a_{1,j}}{a_{2,j}}$ is the unique positive solution to:

$$(cj^2 + b)^{-1} \tau_{i,j}^{-1} \left(\tau_{i,j} - x_j^2 f_{Z,j}^{-1} - f_{D,j}^{-1} \right) = x_j.$$

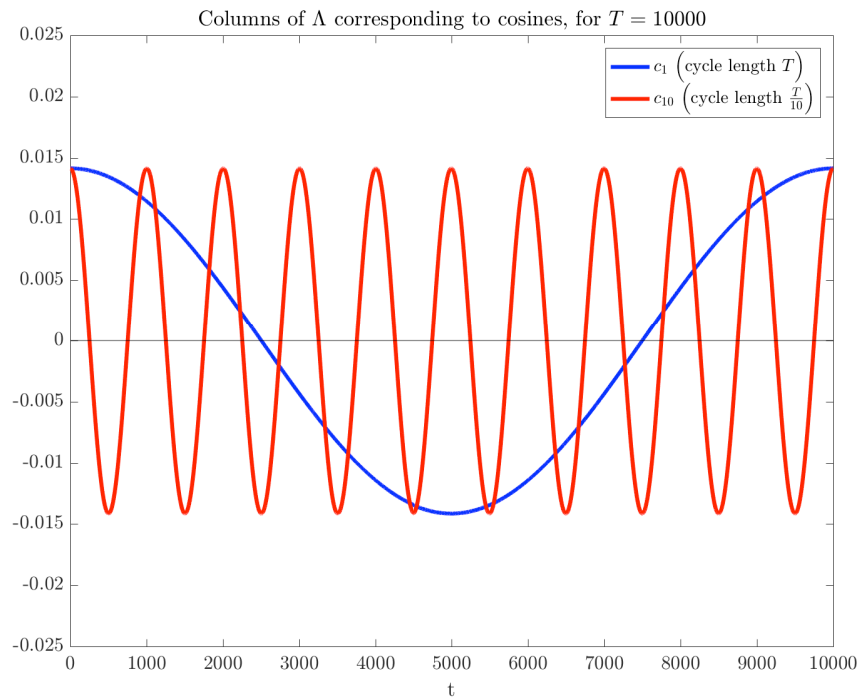


Figure 1: Portfolio weights for the cosing frequency portfolios c_1 and c_{10} , as defined in the main text. The horizontal axis is time, or the maturity of the corresponding futures contract. The vertical axis is the weight which each portfolio puts on that futures contract.

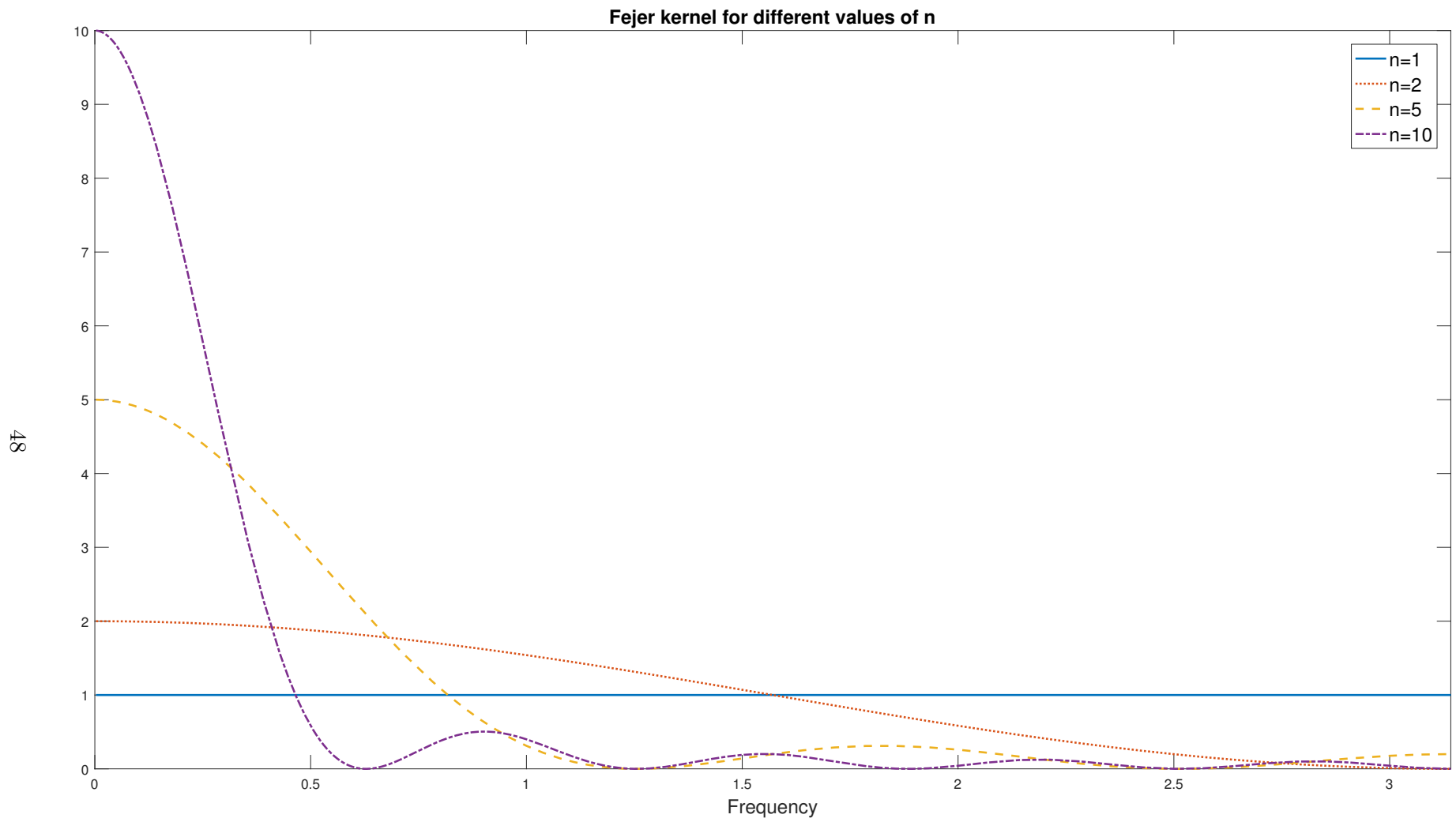


Figure 2: Examples of the Fejér Kernel, for increasing values of n .