Essentially affine approximations for economic models

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Abstract

This paper proposes a novel first-order approximation technique for standard economic models with time-varying risk aversion or volatility. The log-linearization is identical up to the first order to perturbation, but it includes volatility adjustments that perturbation would treat as "higher-order" that follow from the use of closed-form expressions for log-normal expectaions. I calculate Euler equation errors for the RBC model with time-varying risk aversion and volatility and find that the essentially affine approximation has accuracy between that of second and third-order perturbations. The equilibrium dynamics take a fully linear state-space form, so models can be estimated with the Kalman filter, rather than a more computationally intensive nonlinear filter. The approximation encompasses a variety of well-known methods specialized for use in particular settings, including general equilibrium models, models of time-varying risk aversion, portfolio choice, and endowment-economy asset pricing.

Keywords: Approximation, linearization, asset pricing, general equilibrium

1 Introduction

This paper introduces a general method for approximating dynamic economic models that is designed to capture the effects of time-varying volatility and risk aversion in a fully linear setting. In many models, all of the equilibrium equations involving expectations weight future states of the world by a pricing kernel that is identical across equations. That is, they involve so-called risk-neutral expectations over future payoffs. Rather than approximating the risk-neutral expectations directly as in perturbation, I approximate the pricing kernel (the mapping from the physical into the risk-neutral density) and the

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operands of the expectations separately. The key result is that by using the closed-form expression for the expectation of a log-normally distributed variable, we can obtain first-order approximations in which changes in volatility or risk aversion affect the endogenous variables of the model. We can therefore use standard linear filtering methods to estimate dynamic models with time-varying risk premia. This paper provides an approximation framework useful for estimating medium- and large-scale models with realistic descriptions of asset markets.

As a simple example, consider the canonical real business cycle (RBC) model under power utility, in which the Euler equation involves the expectation of the household's stochastic discount factor (SDF) multiplied by the return on capital. The method I propose is to take log-linear approximations to the SDF and return on capital separately. The return on capital, $R_{k,t+1}$, and pricing kernel $M_{t+1} = \beta \left(C_{t+1}/C_t\right)^{-\rho}$ are approximated as

$$\begin{aligned} R_{k,t+1} &\approx \exp\left(f_0 + f_x x_t + f_{x'} x_{t+1}\right) & (1) \\ M_{t+1} &\approx \exp\left(\log\beta - \rho\left(c_{t+1} - c_t\right)\right) & (2) \end{aligned}$$

where x_t is a vector of state variables, c_t is the log of household consumption, and β is the rate of pure time preference.

The Euler equation is then approximated as

$$\begin{aligned} 0 &= \log E_t\left[M_{t+1} R_{k,t+1}\right] & (3) \\ &= \log E_t\left[\exp\left(f_0 + f_x x_t + f_{x'} x_{t+1} + \log\beta - \rho\left(c_{t+1} - c_t\right)\right)\right] & (4) \end{aligned}$$

If x_t and c_t follow linear Gaussian processes, then the right-hand size of (4) will be linear in x_t, $E_t x_{t+1}$, c_t, and $E_t c_{t+1}$, but it will also have an adjustment for the volatilities of x_{t+1} and c_{t+1} and their covariance, i.e., the covariance of the SDF and the return on capital:

$$0 = f_0 + f_x x_t + f_{x'} E_t x_{t+1} + \log\beta - \rho\left(E_t c_{t+1} - c_t\right) + \frac{1}{2} var\left(f_{x'} x_{t+1} - \rho c_{t+1}\right) \quad (5)$$

The first-order approximation thus directly accounts for precautionary-saving and risk-premium effects. An alternative interpretation of the approximation for the RBC model is that if the household actually faced a log-linear return on capital and budget constraint, the solution obtained from the essentially affine approximation would be exact.

The analysis takes advantage of the fact that the pricing kernel in economic models often falls into the essentially affine class described by Duffee (2002), so I call it the essentially affine approximation method. Essentially affine pricing kernels allow for time-varying risk premia, but risk-neutral expectations over payoffs that are log-linear in the state variables will themselves be log-linear in the state variables.

The essentially affine method is designed to accommodate two main sources of time variation in volatility. First, the fundamental shocks in the model may be heteroskedastic. Second, the pricing kernel, even though it is conditionally linear, may have a time-varying loading on the fundamental shocks, for example due to time-varying risk aversion. Critically, and unlike first-order perturbation, the method does not impose or imply certainty equivalence. Movements in risk premia can have real effects on endogenous variables. So, for example, this method makes it possible to use the standard Bayesian methods in the macro literature (e.g., Smets and Wouters, 2003) to estimate the effects of changes in risk premia on output, consumption, and investment.

To allow for shifts in risk premia to have real effects in equilibrium models, the standard current method is to use higher-order perturbation (e.g., Rudebusch and Swanson, 2011). But methods for estimating nonlinear models, such as the particle filter, tend to be orders of magnitude slower than linear methods, which can seriously hinder empirical work (Fernandez-Villaverde and Rubio-Ramirez, 2007). Consider, for example, the problem of estimating a structural model of the term structure of interest rates (e.g., Bekaert, Cho, and Moreno, 2010). It is well known that the likelihood surface in term structure models tends to have many flat points and local maxima, and hence finding the global maximum can require extensive searching. This searching may be infeasibly slow with nonlinear estimation techniques that use simulations to calculate the likelihood.[1] Moreover, higher-order perturbations require the calculation of many more derivatives than the essentially affine method, which can be costly in large models (the required number of derivatives grows exponentially with the order of the approximation). Although the essentially affine method may not ease estimation when the fundamental shocks are heteroskedastic (since a nonlinear filter is required whatever the order of the solution), Dew-Becker (2011a) finds that it is useful for estimating a model with time-varying risk aversion.

[1] In fact, Ang and Piazessi (2003) are unable to find a global maximum in a completely linear term structure model. The term structure literature has thus far generally avoided the added trouble of allowing for nonlinearity.

Even though the essentially affine approximation allows time-varying risk premia to have real effects, it is still based on a set of linearizations. Because we linearize parts of the model, the method is no more than first-order accurate local to the non-stochastic steady state. Specifically, when there is no volatility, the essentially affine approximation is identical to a first-order perturbation (the essentially affine method includes corrections involving σ^2, the variance of the shocks, which perturbation treats as higher-order).

To show that there is a benefit to using the essentially affine method, I calculate Euler equation errors for the RBC model under stochastic volatility and time-varying risk aversion. In both cases, the essentially affine solution is at least an order of magnitude more accurate than a first-order perturbation and has accuracy between that of second- and third-order perturbations. However, the essentially affine method has the advantage over higher-order approximations of being compatible with linear filtering methods, which makes it well suited to empirical analysis.

The macro literature has recently begun to explore the roles of time-varying volatility and risk premia in the business cycle (e.g., Bloom, 2009; Fernandez-Villaverde et al., 2011; Gourio, 2010; Lettau and Uhlig, 2000; Rudebusch and Swanson, 2011; Tallarini, 2000). These papers, however, require the use of either high-order perturbations or more complex so-called global solution methods (e.g., projection) and no so-called medium-scale models have been estimated with time-varying risk premia. Benigno, Benigno, and Nistico (2010) recently suggest a variation on perturbation in which asset prices respond to stochastic volatility in the first-order approximation (see also Jermann, 1998), but movements in volatility do not affect real variables, unlike here.

Malkhozov and Shamloo (2011) study an alternative method that also takes advantage of log-normal formulas. Their method is designed to accommodate stochastic volatility, but they do not focus on using exact formulas for the pricing kernel (the essentially affine case, in particular) as I do. I also show how to approximate models of time-varying risk aversion, which their framework does not accommodate.

Essentially affine methods are widely used in the asset pricing literature, not only for pricing bonds (Duffee, 2002) but also, more recently, for pricing equities (e.g., Bansal and Yaron, 2004; Lettau and Wachter, 2007). Earlier, Campbell and Koo (1997) and Campbell and Viceira (1999), among many others, also employ a special case of the essentially affine method for studying portfolio choice models. This paper gives a general treatment of their methods and provides a rigorous characterization of their local accuracy.

The remainder of this paper is organized as follows. I begin by describing the basic environment in section 2. Next, I move from the general approximation method to the specifics of the first-order method in section 3. As examples, I show how to solve the RBC model with power utility with homoskedastic and heteroskedastic shocks and also Epstein–Zin preferences with time-varying risk aversion, as in Dew-Becker (2011b). The derivations are slightly different depending on whether we use stochastic volatility or time-varying risk aversion, but the output of the approximation takes the same log-linear form for both cases. The final section of the paper examines Euler equation errors and shows that the essentially affine approximation is substantially more accurate that a first-order approximation in the benchmark models, and competitive with higher-order perturbations.

2 Preliminaries

I consider a dynamic system of variables contained in the vector X_t of dimension $N_X \times 1$. There is a vector of mean-zero, normal, and serially uncorrelated exogenous shocks ε_t of dimension N_ε.

The economic model is defined by a set of constraints determining the dynamics of X_t

$$0_{N_X \times 1} = G\left(X_t, X_{t+1}, \sigma \varepsilon_{t+1}\right) \tag{6}$$

where $0_{N_X \times 1}$ is a vector of zeros of length N_X, $G : \mathbb{R}^{N_X} \times \mathbb{R}^{N_X} \times \mathbb{R}^{N_\varepsilon} \rightarrow \mathbb{R}^{N_X}$, and σ is a scalar used to control the variances of the shocks in the approximations. $\sigma = 1$ in the true model, but we will also consider behavior local to $\sigma = 0$, as in the perturbation literature. The function G may (and usually will) involve the expectation operator.[2]

We divide the equations into two types. The first is a set of N_D non-expectational equations that take the form

$$0_{N_D \times 1} = D\left(X_t, X_{t+1}, \sigma \varepsilon_{t+1}\right) \tag{7}$$

with $D : \mathbb{R}^{N_X} \times \mathbb{R}^{N_X} \times \mathbb{R}^{N_\varepsilon} \rightarrow \mathbb{R}^{N_D}$. The defining characteristic of these equations is that they do not involve the expectation operator. D could include, for example, budget constraints.

[2] Malkohozov and Shamloo (2011) make a recent contribution. They study an alternative case in which $0_{N_X \times 1} = f\left(x_{t-1}, x_t, E_t \Gamma \exp x_{t+1}\right)$ for some function f where E_t denotes the expectation operator conditional on information available at date t, which is more restrictive than (6). It is not clear how many economic models can be expressed in the form $0_{N_X \times 1} = f\left(x_{t-1}, x_t, E_t \Gamma \exp x_{t+1}\right)$ (the RBC model studied here cannot). More important, Malkhozov and Shamloo's work only discusses stochastic (and Gaussian) volatility, whereas the analysis here was originally designed to cover time-varying risk aversion.

The second set of equations is forward-looking and involves mathematical expectations,

$$1_{N_F \times 1} = E_t \left[M \left(X_t, X_{t+1}, \sigma \varepsilon_{t+1} \right) \times F \left(X_t, X_{t+1}, \sigma \varepsilon_{t+1} \right) \right] \tag{8}$$

where E_t denotes the expectation operator conditional on information available at date t. The function $F : \mathbb{R}^{N_X} \times \mathbb{R}^{N_X} \times \mathbb{R}^{N_\varepsilon} \rightarrow \mathbb{R}^{N_F}$, with $N_F + N_D = N_X$, is vector-valued, whereas $M : \mathbb{R}^{N_X} \times \mathbb{R}^{N_X} \times \mathbb{R}^{N_\varepsilon} \rightarrow \mathbb{R}$ is a scalar function. The function $M \left(X_t, X_{t+1}, \sigma \varepsilon_{t+1} \right)$ is a pricing kernel or stochastic discount factor (SDF) that reweights the expectation. In the RBC model, for example, $M \left(X_t, X_{t+1}, \sigma \varepsilon_{t+1} \right)$ is the household's marginal rate of substitution between dates t and $t+1$, while F is the gross return on investment.

Example 1: The RBC model with power utility. The representative household maximizes

$$\sum_{j=0}^{\infty} \beta^j \frac{C_{t+j}^{1-\rho}}{1-\rho} \tag{9}$$

subject to the constraints

$$\begin{aligned} C_t + K_{t+1} &= (1-\delta) K_t + A_t K_t^\alpha & (10) \\ \log A_t &= \phi \log A_{t-1} + \sigma \varepsilon_t & (11) \end{aligned}$$

where C_t is consumption, K_t capital, and A_t the level of technology.

The Euler equation is

$$1 = E_t \left[\beta \left(C_{t+1}/C_t \right)^{-\rho} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta \right) \right] \tag{12}$$

In the notation from above, we can say $X_t = [c_t, k_t, a_t]'$, where lowercase letters denote logs, and

$$\begin{aligned} D \left(X_t, X_{t+1}, \sigma \varepsilon_{t+1} \right) &= \begin{bmatrix} (1-\delta) \exp(k_t) + \exp(a_t + \alpha k_t) - \exp(c_t) - \exp(k_{t+1}) \\ a_{t+1} - \phi a_t - \sigma \varepsilon_{t+1} \end{bmatrix} & (13) \\ M \left(X_t, X_{t+1}, \sigma \varepsilon_{t+1} \right) &= \beta \exp\left(-\rho \left(c_{t+1} - c_t \right) \right) & (14) \\ F \left(X_t, X_{t+1}, \sigma \varepsilon_{t+1} \right) &= \alpha \exp\left(a_{t+1} + (\alpha - 1) k_{t+1} \right) + 1 - \delta & (15) \end{aligned}$$

$\blacksquare$

The separation of M and F is the key difference between the essentially affine method and perturbation. The basic observation is simply that in many widely used economic models, the equilibrium equations that involve expectations are weighted by the household's stochastic discount factor. This is true in the simple RBC model, and medium-scale New-Keynesian models (e.g., Smets and Wouters, 2003) generally have the same feature. Even if a model has an expectational equation that does not involve the SDF, i.e., $1 = E_t \left[Z \left(X_t, X_{t+1}, \sigma\varepsilon_{t+1} \right) \right]$ for some function Z, it is always possible to write it in the form (8) simply by multiplying and dividing by M,

$$1 = E_t \left[M \left(X_t, X_{t+1}, \sigma\varepsilon_{t+1} \right) \times \frac{Z \left(X_t, X_{t+1}, \sigma\varepsilon_{t+1} \right)}{M \left(X_t, X_{t+1}, \sigma\varepsilon_{t+1} \right)} \right] \tag{16}$$

and then setting $F = Z/M$ in equation (8).

3 Log-linear solutions

This section derives first-order approximations to the system

$$0_{N_D \times 1} = D \left(X_t, X_{t+1}, \sigma\varepsilon_{t+1} \right) \tag{17}$$

$$1_{N_F \times 1} = E_t \left[M \left(X_t, X_{t+1}, \sigma\varepsilon_{t+1} \right) \times F \left(X_t, X_{t+1}, \sigma\varepsilon_{t+1} \right) \right] \tag{18}$$

I consider two related cases. The first uses log-linear approximation to the pricing kernel M and allows for stochastic volatility. The second is for a so-called essentially affine pricing kernel when risk aversion is time-varying. In both cases, I show that the solutions coincide with standard perturbation approximations except that they include a volatility correction. In the case of a deterministic model, the solutions coincide exactly with perturbation.

The basic idea is to use log-linear approximations for M, D, and F, and then find a process for X_t that solves (17-18) exactly. In the RBC model, this corresponds to using the exact formula for the household's marginal rate of substitution, $\beta \left(C_{t+1}/C_t \right)^{-\rho}$, and log-linearizing the budget constraint and the return on capital. We then solve for the law of motion for capital (equivalently, the consumption function) that jointly satisfies the budget constraint and the Euler equation.

3.1 Steady state and approximations

In both cases of the approximation I consider, the approximations are taken around the non-stochastic steady state, defined as the $\bar{X}$ that solves the system

$$0_{N_D \times 1} = D\left(\bar{X}, \bar{X}, 0\right) \tag{19}$$

$$1_{N_F \times 1} = E_t\left[M\left(\bar{X}, \bar{X}, 0\right) \times F\left(\bar{X}, \bar{X}, 0\right)\right] \tag{20}$$

We will repeatedly use first-order approximations to the various functions, so define

$$D_X \equiv \frac{\partial D\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)}{\partial X_t} \tag{21}$$

$$D_{X'} \equiv \frac{\partial D\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)}{\partial X_{t+1}} \tag{22}$$

$$D_{\varepsilon} \equiv \frac{\partial D\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)}{\partial \varepsilon_{t+1}} \tag{23}$$

where the derivatives here and throughout are evaluated at $\left(\bar{X}, \bar{X}, 0\right)$. The derivatives of M and F are denoted analogously. We will use using log-linear approximations to M and F, so we define

$$m\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) \equiv \log\left(M\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)\right) \tag{24}$$

$$f\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) \equiv \log\left(F\left(X_t, X_{t+1}, \varepsilon_{t+1}\right)\right) \tag{25}$$

with the derivatives of m and f denoted as above. Also,

$$\bar{D} \equiv D\left(\bar{X}, \bar{X}, 0\right) \tag{26}$$

and $\bar{M}$ and $\bar{F}$ are defined analogously. Last, a circumflex denotes the deviation from steady-state,

$$\hat{X}_t \equiv X_t - \bar{X} \tag{27}$$

3.2 Case 1: Constant volatility and a linearized SDF

Suppose the shocks are homoskedastic and normally distributed:

$$\varepsilon_t \sim \mathcal{N}\left(0_{N_\varepsilon \times 1}, \Sigma\right) \tag{28}$$

We approximate the three functions determining the equilibrium as

$$\begin{aligned} D_{t+1}^{(1)} &= \bar{D} + D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \sigma \varepsilon_{t+1} && (29) \\ M_{t+1}^{(1)} &= \exp\left(\bar{m} + m_X \hat{X}_t + m_{X'} \hat{X}_{t+1} + m_\varepsilon \sigma \varepsilon_{t+1}\right) && (30) \\ F_{t+1}^{(1)} &= \exp\left(\bar{f} + f_X \hat{X}_t + f_{X'} \hat{X}_{t+1} + f_\varepsilon \sigma \varepsilon_{t+1}\right) && (31) \end{aligned}$$

where the superscript (1) denotes a first-order approximation. The approximations to the equilibrium conditions (17–18) are

$$\begin{aligned} 0 &= D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \sigma \varepsilon_{t+1} && (32) \\ 0 &= \log E_t \exp\left(\left(m_X + f_X\right) \hat{X}_t + \left(m_{X'} + f_{X'}\right) \hat{X}_{t+1} + \left(m_\varepsilon + f_\varepsilon\right) \sigma \varepsilon_{t+1}\right) && (33) \end{aligned}$$

(where $\exp\left(\bar{m} + \bar{f}\right) = 1$ and $\bar{D} = 0$ from the definition of the non-stochastic steady-state).

Now we guess that the solution to this system takes the form

$$\hat{X}_{t+1} = H_0 + H_X \hat{X}_t + H_\varepsilon \sigma \varepsilon_t \tag{34}$$

Plugging this guess into (32–33), we obtain

$$0 = D_X \hat{X}_t + D_{X'} \left(H_0 + H_X \hat{X}_t + H_\varepsilon \sigma \varepsilon_t\right) + D_\varepsilon \sigma \varepsilon_{t+1} \tag{35}$$

and

$$0 = \log E_t \exp \begin{pmatrix} (m_X + f_X)\hat{X}_t \\ +(m_{X'} + f_{X'})\left(H_0 + H_X \hat{X}_t + H_\varepsilon \varepsilon_t\right) + (m_\varepsilon + f_\varepsilon)\sigma\varepsilon_{t+1} \end{pmatrix} \tag{36}$$

$$= (m_X + f_X)\hat{X}_t + (m_{X'} + f_{X'}) E_t \hat{X}_{t+1}$$
$$+\frac{1}{2}\sigma^2 (m_\varepsilon + f_\varepsilon + (m_{X'} + f_{X'}) H_\varepsilon) \Sigma (m_\varepsilon + f_\varepsilon + (m_{X'} + f_{X'}) H_\varepsilon)' \tag{37}$$

where (37) uses the formula for the expectation of a log-normally distributed variable.

So the goal is to find matrices H_0, H_X, and H_ε that solve (35) and (37). Sims (2001) provides an algorithm, Gensys, that solves systems of equations taking the form of (35) and (37).[3] The Gensys algorithm will give values for H_0, H_X, and H_ε, but we have the problem that the system of equations involves the as-yet unknown matrix H_ε. We therefore solve the model iteratively. The full solution algorithm is as follows:

Solution Algorithm: Basic case

Step 1: Approximate the three components of the equilibrium conditions as in (29-31).

Step 2: Use the Gensys algorithm to solve the system

$$0 = (f_X + m_X)\hat{X}_t + (f_{X'} + m_{X'}) E_t \hat{X}_{t+1} \tag{38}$$

$$0 = D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \sigma \varepsilon_{t+1} \tag{39}$$

at $\sigma = 1$, which delivers coefficient matrices $H_0^{(0)}$, $H_X^{(0)}$, and $H_\varepsilon^{(0)}$ for the a law of motion for $\hat{X}_t$,

$$\hat{X}_{t+1} = H_0^{(0)} + H_X^{(0)} \hat{X}_t + H_\varepsilon^{(0)} \sigma \varepsilon_{t+1} \tag{40}$$

Step 3: For $j > 0$, solve the system (again, at $\sigma = 1$)

$$0 = (f_X + m_X)\hat{X}_t + (f_{X'} + m_{X'}) E_t \hat{X}_{t+1}$$
$$+\frac{1}{2}\sigma^2 \left((f_{X'} + m_{X'}) H_\varepsilon^{(j-1)} + f_\varepsilon + m_\varepsilon\right) \Sigma \left((f_{X'} + m_{X'}) H_\varepsilon^{(j-1)} + f_\varepsilon + m_\varepsilon\right)' \tag{41}$$

$$0 = D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \sigma \varepsilon_{t+1} \tag{42}$$

[3] Code for Gensys, is available on his website, http://www.princeton.edu/~sims/

Iterate on step 3 until the matrices $H_0^{(j)}$, $H_X^{(j)}$, $H_\sigma^{(j)}$, and $H_\varepsilon^{(j)}$ converge.

■

The first thing to note about the solution method is that it requires an iteration. This is because the effect of volatility on the equilibrium conditions, which appears through the term $\frac{1}{2}\sigma^2\left(\left(f_{X'}+m_{X'}\right)H_\varepsilon+f_\varepsilon+m_\varepsilon\right)\Sigma\left(\left(f_{X'}+m_{X'}\right)H_\varepsilon+f_\varepsilon+m_\varepsilon\right)'$ in equation (37), depends on the equilibrium dynamics of the model through the H_ε matrix. In general, the iteration method proposed here need not converge. However, if it does, we clearly have a valid solution to the model.[4]

I show below that if we stopped at step 2, we would have a standard perturbation solution to the model. The essentially affine method differs from perturbation because it corrects for the volatility term. The volatility term represents a discount for risk in the Euler equations that first-order perturbation ignores. It means, for example, that in the RBC model, volatility will affect investment through its effects on risk premia and precautionary saving.

The example in section 2 was a general-equilibrium model, but the analysis here also covers methods widely used in the asset-pricing literature. Bansal and Yaron's (2004) analysis of economies with long-run risks and the approximations used for the analysis of portfolio choice problems, e.g. Campbell and Koo (1997) and Campbell and Viceira (1999), are both special cases of the essentially affine method.

Example 1 (continued): To approximate the RBC model, we approximate the three functions D, m, and f. Recall that $X_t=[c_t,k_t,a_t]'$,

$$D\left(X_t,X_{t+1},\sigma\varepsilon_{t+1}\right) = \begin{bmatrix} (1-\delta)\exp\left(k_t\right)+\exp\left(a_t+\alpha K_t\right)-\exp\left(c_t\right)-\exp\left(k_{t+1}\right) \\ a_{t+1}-\phi a_t-\sigma\mu_{t+1} \end{bmatrix} \tag{43}$$

$$M\left(X_t,X_{t+1},\sigma\varepsilon_{t+1}\right) = \beta\exp\left(-\rho\left(c_{t+1}-c_t\right)\right) \tag{44}$$

$$F\left(X_t,X_{t+1},\sigma\varepsilon_{t+1}\right) = \alpha\exp\left(a_{t+1}+(\alpha-1)\,k_{t+1}\right)+1-\delta \tag{45}$$

In the notation from above,

$$\begin{gathered} m_0=\log\beta \qquad m_X=[\rho,0,0] \\ m_{X'}=[-\rho,0,0] \qquad m_\varepsilon=0 \end{gathered} \tag{46}$$

[4] In both simple and complex models, I find convergence to be rapid, generally occuring in fewer than 10 iterations. I encounter failures to converge only in cases with extreme parameter values.

We take a log-linear approximation to F to obtain

$$\begin{aligned} f\left(X_t, X_{t+1}, \sigma\varepsilon_{t+1}\right) &\approx \log\left(\alpha \exp\left(\bar{a}\right)\exp\left(\left(\alpha-1\right)\bar{k}\right)+1-\delta\right) \\ &\quad +\tfrac{\alpha\bar{A}\bar{K}^{\alpha-1}}{\alpha\bar{A}\bar{K}^{\alpha-1}+1-\delta}\left(\hat{a}_{t+1}+\left(\alpha-1\right)\hat{k}_{t+1}\right) \end{aligned} \tag{47}$$

And hence

$$f_0 = \log\left(\alpha \exp\left(\bar{a}\right)\exp\left(\left(\alpha-1\right)\bar{k}\right)+1-\delta\right) \tag{48}$$

$$f_X = [0,0,0] \tag{49}$$

$$f_{X'} = \left[0, \tfrac{\alpha(\alpha-1)\bar{A}\bar{K}^{\alpha-1}}{\alpha\bar{A}\bar{K}^{\alpha-1}+1-\delta}, \tfrac{\alpha\bar{A}\bar{K}^{\alpha-1}}{\alpha\bar{A}\bar{K}^{\alpha-1}+1-\delta}\right] \tag{50}$$

$$f_\varepsilon = 0 \tag{51}$$

Finally, we approximate D as

$$D\left(X_t, X_{t+1}, \sigma\varepsilon_{t+1}\right) \approx \begin{bmatrix} \left(1-\delta\right)\exp\left(\bar{k}\right)\hat{k}_t + \exp\left(\bar{a}+\alpha\bar{k}\right)\left(\hat{a}_t+\alpha\hat{k}_t\right) - \exp\left(\bar{c}\right)\hat{c}_t - \exp\left(\bar{k}\right)\hat{k}_{t+1} \\ a_{t+1}-\phi a_t - \varepsilon_{t+1} \end{bmatrix} \tag{52}$$

and

$$D_X = \begin{bmatrix} 0 & -\exp\left(\bar{k}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{53}$$

$$D_{X'} = \begin{bmatrix} -\exp\left(\bar{c}\right) & \left(1-\delta\right)\exp\left(\bar{k}\right)+\alpha\exp\left(\bar{a}+\alpha\bar{k}\right) & \exp\left(\bar{a}+\alpha\bar{k}\right) \\ 0 & 0 & -\phi \end{bmatrix} \tag{54}$$

$$D_{\varepsilon'} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{55}$$

■

Note that in this section we are using the exact formula for the SDF – there is no approximation. So the solution to the RBC model here can be viewed as an exact solution to an economy in which the household faces a fundamentally log-linear budget constraint and return on investment.[5]

[5] The solution here can be shown to coincide exactly with that of Campbell (1994) if his log-linear approximations are also

More generally, the solution algorithm in this case does not require that the SDF is exactly log-linear even though constant relative risk aversion tends to induce a log-linear SDF (e.g., under power utility and Epstein–Zin preferences), so an approximation to the SDF will often not be required. That is, $m = m_0 + m_x \hat{X}_t + m_{x'} \hat{X}_{t+1}$ exactly in most cases.

3.3 Case 1a: Stochastic volatility and a linearized SDF

Given the analysis above, it is straightforward to accommodate stochastic volatility. Suppose the conditional distribution of ε_t is

$$\varepsilon_t \sim \mathcal{N}\left(0_{N_\varepsilon \times 1}, \Sigma_t\right) \tag{56}$$

Σ_t is an $N_\varepsilon \times N_\varepsilon$ variance matrix for ε_t with elements, $vec\left(\Sigma_t\right)$, that are contained in the vector of endogenous variables X_t, (where $vec\left(\cdot\right)$ is the vectorization operator that stacks the columns of a matrix). Note that the dynamics of $vec\left(\Sigma_t\right)$ cannot be completely unrestricted since Σ_t must remain positive semi-definite.

The equilibrium conditions (35) and (37) then become

$$\begin{aligned} 0 &= \left(m_X + f_X\right) \hat{X}_t + \left(m_{X'} + f_{X'}\right) E_t \hat{X}_{t+1} \\ &\quad + \frac{1}{2}\sigma^2 \left(m_\varepsilon + f_\varepsilon + \left(m_{X'} + f_{X'}\right) H_\varepsilon\right) \Sigma_t \left(m_\varepsilon + f_\varepsilon + \left(m_{X'} + f_{X'}\right) H_\varepsilon\right)' \qquad (57) \\ 0 &= D_X \hat{X}_t + D_{X'} \left(H_0 + H_X \hat{X}_t + H_\varepsilon \sigma \varepsilon_t\right) + D_\varepsilon \sigma \varepsilon_{t+1} \qquad (58) \end{aligned}$$

The only difference is that there is now a time subscript on Σ_t. But because all the elements of Σ_t are contained in X_t, $\frac{1}{2}\sigma^2 \left(m_\varepsilon + f_\varepsilon + \left(m_{X'} + f_{X'}\right) H_\varepsilon\right) \Sigma_t \left(m_\varepsilon + f_\varepsilon + \left(m_{X'} + f_{X'}\right) H_\varepsilon\right)'$ is linear in $\hat{X}_t$, and we can use the same solution algorithm as in case 1.

3.4 Case 2: Time-varying risk aversion

In this case, we allow for the possibility of time-varying risk aversion, which generates endogenous heteroskedasticity in the SDF. This case encompasses Epstein–Zin preferences both in their standard form (Epstein and Zin, 1991) and with time-varying risk aversion, as in Melino and Yang (2003) and Dew-Becker

taken at the non-stochastic steady-state. Lettau (2003) extends Campbell's (1993) analysis to cover the case of Epstein–Zin preferences. Case 2 below covers his analysis.

(2011b). Specifically, the Epstein–Zin SDF with time-varying risk aversion is

$$M_{t+1} = \exp\left(\frac{1-\alpha_t}{1-\rho}\log\beta - \rho\frac{1-\alpha_t}{1-\rho}\left(c_{t+1}-c_t\right) + \frac{\rho-\alpha_t}{1-\rho}r_{w,t+1}\right) \tag{59}$$

where r_w is the log return on the household's wealth (i.e., a claim to its consumption stream).

Assume that the shocks are normal and homoskedastic:

$$\varepsilon_t \sim \mathcal{N}\left(0_{N_\varepsilon \times 1}, \Sigma\right) \tag{60}$$

Given (59), we need to solve the Euler equations not only for F (e.g. the return on capital), but also for the return on the household's total wealth. The two equations are

$$\begin{aligned} 1 &= E_t\left[\exp\left(m_{t+1}\right) F\left(X_t, X_{t+1}, \sigma\varepsilon_{t+1}\right)\right] && (61) \\ 1 &= E_t\left[\exp\left(m_{t+1} + r_{w,t+1}\right)\right] && (62) \end{aligned}$$

To solve the model, we take a first-order approximation to F as before, giving

$$\begin{aligned} 0 &= \log E_t\left[\exp\left(\begin{array}{c} \frac{1-\alpha_t}{1-\rho}\log\beta - \rho\frac{1-\alpha_t}{1-\rho}\left(c_{t+1}-c_t\right) + \frac{\rho-\alpha_t}{1-\rho}r_{w,t+1} \\ +f_0 + f_X\hat{X}_t + f_{X'}\hat{X}_{t+1} + f_\varepsilon\sigma\varepsilon_{t+1} \end{array}\right)\right] && (63) \\ &= \log E_t\left[\exp\left(\begin{array}{c} -\rho\frac{1-\alpha_t}{1-\rho}\Gamma_c\left(\hat{X}_{t+1}-\hat{X}_t\right) + \frac{\rho-\alpha_t}{1-\rho}\Gamma_w\hat{X}_{t+1} \\ +f_X\hat{X}_t + f_{X'}\hat{X}_{t+1} + f_\varepsilon\sigma\varepsilon_{t+1} \end{array}\right)\right] && (64) \end{aligned}$$

where Γ_c and Γ_w are selection vectors such that $\Gamma_c X_t = c_t$ and $\Gamma_w X_t = r_{w,t+1}$. Now we guess that

$$\hat{X}_{t+1} = H_0 + H_X\hat{X}_t + H_\varepsilon\sigma\varepsilon_{t+1} \tag{65}$$

Substituting (65) into (64) yields

$$\begin{aligned} 0 \;=\; & -\rho\frac{1-\alpha_t}{1-\rho}\Gamma_c\left(E_t\hat{X}_{t+1}-\hat{X}_t\right) + \frac{\rho-\alpha_t}{1-\rho}\Gamma_w E_t\hat{X}_{t+1} + f_X\hat{X}_t + f_{X'}E_t\hat{X}_{t+1} \\ & +\frac{1}{2}\sigma^2\left(-\rho\frac{1-\alpha_t}{1-\rho}\Gamma_c + \frac{\rho-\alpha_t}{1-\rho}\Gamma_w + f_{X'} + f_\varepsilon\right)H_\varepsilon\Sigma H_\varepsilon'\left(-\rho\frac{1-\alpha_t}{1-\rho}\Gamma_c + \frac{\rho-\alpha_t}{1-\rho}\Gamma_w + f_{X'} + f_\varepsilon\right)' && (66) \end{aligned}$$

Similarly, for the wealth portfolio, the Euler equation (62) becomes

$$\begin{aligned} 0 &= \log E_t \exp\left(-\rho \frac{1-\alpha_t}{1-\rho} \Gamma_c \left(\hat{X}_{t+1} - \hat{X}_t\right) + \frac{1-\alpha_t}{1-\rho} \Gamma_w \hat{X}_{t+1}\right) && (67) \\ &= \left(-\rho \Gamma_c + \Gamma_w\right) \left(E_t \hat{X}_{t+1} - \hat{X}_t\right) + \frac{1}{2} \frac{1-\alpha_t}{1-\rho} \sigma^2 \left(-\rho \Gamma_c + \Gamma_w\right) H_\varepsilon \Sigma H_\varepsilon' \left(-\rho \Gamma_c + \Gamma_w\right)' && (68) \end{aligned}$$

Substituting (68) into (66) yields

$$\begin{aligned} 0 = & -\Gamma_w E_t \hat{X}_{t+1} + f_X \hat{X}_t + f_{X'} E_t \hat{X}_{t+1} \\ & + \frac{1 - \bar{\alpha} - \Gamma_\alpha \hat{X}_t}{1-\rho} \left(-\Gamma_w + f_{X'} + f_\varepsilon\right) H_\varepsilon \Sigma H_\varepsilon' \left(-\rho \Gamma_c + \Gamma_w\right)' \\ & + \frac{1}{2} \sigma^2 \left(-\Gamma_w + f_{X'} + f_\varepsilon\right) H_\varepsilon \Sigma H_\varepsilon' \left(-\Gamma_w + f_{X'}\right)' \qquad (69) \end{aligned}$$

where Γ_α is the selection vector such that $\Gamma_\alpha X_t = \alpha_t$.

Looking at (66), the Jensen adjustment on the second line is quadratic in one of the state variables, α_t. This is part of the time-varying precautionary saving effect that arises due to movements in risk aversion. The same effect appears in the Euler equation for the wealth portfolio, (68), so we can use (68) to substitute the quadratic terms out of (66).

Finally, (69) is, once again, linear in the state variables and can be solved using standard methods. As in case 1, though, the system of equations depends on the decision rule through H_ε. So again we use an iteration. The algorithm is as follows:

Solution Algorithm: Epstein–Zin preferences with time-varying risk aversion

Step 1: Linearly approximate $D\left(x_t, X_{t+1}, \sigma\varepsilon_{t+1}\right)$ and $f\left(x_t, X_{t+1}, \sigma\varepsilon_{t+1}\right)$

Step 2: Use the Gensys algorithm to solve the system (at $\sigma = 1$)

$$\begin{aligned} 0 &= f_X + \hat{X}_t + \left(f_{X'} - \Gamma_w\right) E_t \hat{X}_{t+1} && (70) \\ 0 &= D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \sigma \varepsilon_{t+1} && (71) \end{aligned}$$

which delivers a law of motion for $\hat{X}_t$

$$\hat{X}_{t+1} = H_0^{(0)} + H_X^{(0)} \hat{X}_t + H_\varepsilon^{(0)} \sigma \varepsilon_{t+1} \qquad (72)$$

Step 3: For $j > 0$, solve the system

$$
\begin{aligned}
0 &= -\Gamma_w E_t \hat{X}_{t+1} + f_X \hat{X}_t + f_{X'} E_t \hat{X}_{t+1} \\
&\quad + \sigma^2 \frac{1 - \bar{\alpha} - \Gamma_\alpha \hat{X}_t}{1-\rho} \left(-\Gamma_w + f_{X'} + f_\varepsilon\right) H_\varepsilon \Sigma H_\varepsilon' \left(-\rho \Gamma_c + \Gamma_w\right)' && (73) \\
&\quad + \sigma^2 \frac{1}{2} \left(-\Gamma_w + f_{X'} + f_\varepsilon\right) H_\varepsilon \Sigma H_\varepsilon' \left(-\Gamma_w + f_{X'}\right)' && (74) \\
0 &= D_X \hat{X}_t + D_{X'} \hat{X}_{t+1} + D_\varepsilon \sigma \varepsilon_{t+1} && (75)
\end{aligned}
$$

Iterate on step 3 until the matrices $H_0^{(j)}$, $H_X^{(j)}$, $H_\sigma^{(j)}$, and $H_\varepsilon^{(j)}$ converge.

■

Now note that we have not used any approximation to the SDF here. The only equations we approximate are those for F and D.

Example 2: RBC model with Epstein–Zin preferences and time-varying risk aversion

The household's objective function is

$$V_t = \left\{ (1-\beta) C_t^{1-\rho} + \beta E_t \left[V_{t+1}^{1-\alpha_t} \right]^{\frac{1-\rho}{1-\alpha_t}} \right\}^{\frac{1}{1-\rho}} \tag{76}$$

where risk aversion follows an exogenous process,

$$\alpha_t = (1 - \phi_\alpha) \bar{\alpha} + \phi_\alpha \alpha_{t-1} + \sigma \varepsilon_{\alpha,t} \tag{77}$$

The budget constraint and technology process are the same as in example 1,

$$
\begin{aligned}
C_t + K_{t+1} &= (1-\delta) K_t + A_t K_t^\alpha && (78) \\
\log A_t &= \phi \log A_{t-1} + \sigma \varepsilon_{A,t} && (79)
\end{aligned}
$$

The SDF is

$$M_{t+1} = \beta^{\frac{1-\alpha_t}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\alpha_t}{1-\rho}} R_{w,t+1}^{\frac{\rho - \alpha_t}{1-\rho}} \tag{80}$$

where $R_{w,t+1}$ is the return on an asset that pays C_t as its dividend. We define the price/dividend ratio on that asset, PC_t, as an auxiliary variable, with

$$R_{w,t+1} = \frac{PC_{t+1}+1}{PC_t}\frac{C_{t+1}}{C_t} \tag{81}$$

The vector X_t is then $X_t = [c_t, k_t, r_{w,t}, pc_t, a_t, \alpha_t]'$, and the equations involved in the equilibrium are

$$D\left(X_t, X_{t+1}, \sigma\varepsilon_{t+1}\right) = \begin{bmatrix} (1-\delta)\exp\left(k_t\right) + \exp\left(a_t + \alpha K_t\right) - \exp\left(c_t\right) - \exp\left(k_{t+1}\right) \\ r_{w,t+1} - \frac{\exp(pc_{t+1})+1}{\exp(pc_t)}\exp\left(c_{t+1} - c_t\right) \\ a_{t+1} - \phi a_t - \sigma\varepsilon_{A,t+1} \\ \alpha_{t+1} - \left(1-\phi_\alpha\right)\bar{\alpha} - \phi_\alpha \alpha_t - \sigma\varepsilon_{\alpha,t+1} \end{bmatrix} \tag{82}$$

$$M\left(X_t, X_{t+1}, \sigma\varepsilon_{t+1}\right) = \beta\exp\left(-\rho\frac{1-\alpha_t}{1-\rho}\left(c_{t+1}-c_t\right) + \frac{\rho-\alpha_t}{1-\rho}r_{w,t+1}\right) \tag{83}$$

$$F\left(X_t, X_{t+1}, \sigma\varepsilon_{t+1}\right) = \alpha\exp\left(a_{t+1} + \left(\alpha-1\right)k_{t+1}\right) + 1 - \delta \tag{84}$$

Now note that M is no longer log-linear in the endogenous variables: α_t interacts with c_t, c_{t+1}, and $r_{w,t+1}$.

As in case 1, we are using the exact formula for the SDF. So the law of motion we obtain for $\hat{X}_t$ can be viewed as an exact solution to a version of the model in which the household faces a log-linear budget constraint and log-linear returns on capital and wealth.

■

3.5 Relationship with perturbation

The standard perturbation solution proceeds by linearizing the product $M \times F$, instead of approximating them separately. Specifically, define

$$J\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) \equiv M\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) \times F\left(X_t, X_{t+1}, \varepsilon_{t+1}\right) - 1 \tag{85}$$

so that the basic equilibrium conditions for the model with homoskedastic shocks and a log-linear SDF, (17–18), are

$$
\begin{aligned}
0 &= E_t \left[J \left(X_t, X_{t+1}, \varepsilon_{t+1} \right) \right] && (86) \\
0 &= D \left(X_t, X_{t+1}, \varepsilon_{t+1} \right) && (87)
\end{aligned}
$$

The approximation to D will be the same in perturbation and the essentially affine approximation. Taking a first-order approximation to J yields,

$$
0 = \left(M_X \bar{F} + F_X \bar{M} \right) \hat{X}_t + \left(M_{X'} \bar{F} + F_{X'} \bar{M} \right) E_t \hat{X}_{t+1} \tag{88}
$$

Now for case 1 with stochastic volatility in which we used a log-linear approximation to the SDF, we have

$$
\begin{aligned}
m_X &\equiv \frac{\partial \log M \left(X_t, X_{t+1}, \varepsilon_{t+1} \right)}{\partial X_t} && (89) \\
&= M_X / \bar{M} && (90)
\end{aligned}
$$

and similar formulas for $m_{X'}$, f_X, and $f_{X'}$. So the equilibrium condition for perturbation (88) can be written as

$$
0 = \left(m_X + f_X \right) \hat{X}_t + \left(m_{X'} + f_{X'} \right) E_t \hat{X}_{t+1} \tag{91}
$$

(91) is identical to the equilibrium condition (37) in cases 1 and 1a when $\sigma = 0$. In other words, they are the same up to a volatility adjustment. Moreover, they are first-order equivalent in σ. Only for second-order changes in σ do the conditions differ. That is the sense in which perturbation and the essentially affine approximation are equivalent up to the first order. The appendix derives the same result for case 2 with time-varying risk aversion.

4 Accuracy of the approximation

The essentially affine approximation is identical to a first-order perturbation local to the non-stochastic steady-state. To see how they differ in a stochastic setting, I calculate Euler equation errors over simulated paths for the state variables. I also report impulse-response functions from two models and show that the essentially affine solution gives results highly similar to higher-order solution methods.

The two examples I consider are identical except that in one case volatility varies over time, whereas risk aversion varies in the other. I use Epstein–Zin preferences in both examples, and the production side of the model is a simple RBC setup.

4.1 Stochastic volatility

I study the simple RBC model as in the previous examples, with Epstein–Zin preferences and stochastic volatility. The household's objective function is

$$V_t = \left\{ (1-\beta) C_t^{1-\rho} + \beta \left(E_t V_{t+1}^{1-\alpha} \right)^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}} \tag{92}$$

and the equilibrium conditions defining the model are

$$\begin{aligned} 0 &= (1-\delta) K_t + A_{t+1}^{1-\gamma} K_t^{\gamma} - C_{t+1} - K_{t+1} && (93) \\ 0 &= \log A_{t+1} - \log A_t - \varepsilon_{a,t+1} && (94) \\ 0 &= \frac{PC_{t+1}+1}{PC_t} \frac{C_{t+1}}{C_t} - R_{W,t+1} && (95) \\ 0 &= s_{t+1} - (1-\phi_s) - \phi_s s_t + \varepsilon_{s,t+1} && (96) \end{aligned}$$

$$\begin{aligned} 1 &= E_t \left[\beta^{\frac{1-\alpha}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\alpha}{1-\rho}} R_{W,t+1}^{\frac{\rho-\alpha}{1-\rho}} \left(\gamma A_{t+1}^{1-\gamma} K_t^{\gamma-1} + 1 - \delta \right) \right] && (97) \\ 1 &= E_t \left[\beta^{\frac{1-\alpha}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\alpha}{1-\rho}} R_{W,t+1}^{\frac{1-\alpha}{1-\rho}} \right] && (98) \end{aligned}$$

The shocks are distributed as

$$\varepsilon_t \equiv \begin{bmatrix} \varepsilon_{a,t} \\ \varepsilon_{s,t} \end{bmatrix} \sim \mathcal{N} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} s_t \sigma_a^2 & 0 \\ 0 & \sigma_s^2 \end{bmatrix} \right) \tag{99}$$

s_t scales the volatility of the productivity shock and has a steady-state of 1. The parameter choices are listed in table 1. σ_s^2 and ϕ_s are chosen so that the standard deviation of s_t is roughly 0.4, similar to Fernandez-Villaverde and Rubio-Ramirez (2010). The other parameters are standard in the literature. Risk aversion is set to 15, and the EIS to 1.5.[6]

I define the Euler equation error as the percentage error in pricing a claim on capital, $1 - E_t\left[M_{t+1}\left(\gamma A_{t+1}^{1-\gamma} K_t^{\gamma-1} + 1 - \delta\right)\right]$. A value of 10^{-1}, for example, means that the model implies households misprice a unit of capital by 10 percent. The appendix discusses the details of how the Euler equation errors are calculated.

Figure 1 plots histograms of the $\log_{10}$ Euler equation errors under the essentially affine approximation and first-, second-, and third-order perturbations. As expected, increasing the order of the perturbation reduces the errors substantially. The essentially affine method has accuracy essentially identical to that of a second-order perturbation. These two approximations have different strengths, though. The second-order perturbation is more faithful to the true budget constraint and return on capital, but it does not allow consumption to respond in any way to innovations to volatility. The essentially affine approximation, on the other hand, does allow a response of consumption to volatility changes. To the extent that our primary interest in a model of this sort is how the economy responds to volatility shocks, the essentially affine method seems preferable to a second-order perturbation.

The choice between a third-order perturbation and the essentially affine method here depends on the researcher's goals. If speed in calculating the model solution itself is important, then the essentially affine method is preferable because it only requires taking one set of derivatives and solving a linear system of equations. The speed advantage of the essentially affine method scales exponentially with the size of the model. For small models in which speed is less critical, the third-order perturbation will be preferable because it is more precise, capturing both the nonlinearities in the model and the response of consumption to volatility.

4.2 Time-varying risk aversion

I now consider the RBC model with constant volatility but time-varying risk aversion as in example 2 above. I parameterize the model similarly to Dew-Becker (2011a); see table 1. Average risk aversion is set to 15, while its unconditional standard deviation is 5.3. Figure 2 plots the $\log_{10}$ Euler equation errors for the same four approximations as in figure 1. In this case, the essentially affine approximation is

[6]Note that becuase technology follows a random walk here, the variables must be rescaled in terms of A_t so that they have a non-stochastic steady-state. The standard transformations apply here.

clearly more accurate than first- and second-order perturbations, but again less accurate than a third-order perturbation. The essentially affine approximation outperforms the first-order perturbation by two orders of magnitude on average and the second-order perturbation by one order of magnitude.

To see how the essentially affine approximation affects the approximated dynamics of the RBC model, figure 3 plots impulse response functions for the RBC model with time-varying risk aversion. Each line gives the response (in percentage points) to a unit standard deviation decline in the coefficient of relative risk aversion. The top panel shows that, as expected, neither the first- nor the second-order perturbation allows a consumption response. The lines for third-order perturbation and the essentially affine approximation show that consumption rises by 0.065 and 0.055 percentage points, respectively. The subsequent dynamics are also similar.

The impulse responses for the expected excess return on the household's wealth are even more similar in the third-order and essentially affine approximations. Both imply that the expected excess return on wealth falls by 0.002 percentage points following the shock (this effect is magnified for levered portfolios), and again the dynamics are essentially identical. As before, the first- and second-order approximations give no response to the risk aversion shock. Figure 3 thus shows that the essentially affine approximation gives qualitatively and quantitatively similar responses to risk aversion shocks as a third-order perturbation.

As with stochastic volatility, third-order perturbation outperforms the essentially affine method in terms of Euler equation errors. In this case, though, the essentially affine method has a key feature that recommends it over higher-order approximations: models can be estimated using the standard Kalman filter. Unlike with stochastic volatility, under time-varying risk aversion, the dynamics of the model under the essentially affine approximation take on the state-space form required for the Kalman filter. So with time-varying risk aversion, the essentially affine approximation enables fast filtering with standard linear methods.

5 Conclusion

This paper introduces a general approximation technique that delivers linear approximations that account for the effects of time-varying risk aversion, volatility, and disaster risk. In canonical settings, the approximation is orders of magnitude more accurate than a first-order perturbation and competitive with second- and third-order perturbations. The method is particularly useful for allowing linear likelihood-based estimation of equilibrium asset pricing models with time-varying risk prices or time-varying disaster risk.

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A Relation with perturbation under time-varying risk aversion

Under perturbation, the approximation to the two Euler equations for Epstein–Zin preferences (61–62) is

$$0 = E_t \left[\begin{array}{c} F_X \bar{M} \hat{X}_t + F_{X'} \bar{M} \hat{X}_{t+1} + F_\varepsilon \bar{M} \varepsilon_{t+1} \\ + \bar{F} \bar{M} \rho \frac{1-\bar{\alpha}}{1-\rho} \Gamma_c \hat{X}_t + \bar{F} \bar{M} \left(-\rho \frac{1-\bar{\alpha}}{1-\rho} \Gamma_c + \frac{\rho - \bar{\alpha}}{1-\rho} \Gamma_w \right) \hat{X}_{t+1} \end{array} \right] \tag{100}$$

$$0 = \bar{M} F_X \hat{X}_t + \bar{M} F_{X'} E_t \hat{X}_{t+1} + \rho \frac{1-\bar{\alpha}}{1-\rho} \Gamma_c \hat{X}_t + \left(-\rho \frac{1-\bar{\alpha}}{1-\rho} \Gamma_c + \frac{\rho - \bar{\alpha}}{1-\rho} \Gamma_w \right) E_t \hat{X}_{t+1} \tag{101}$$

(noting that $\bar{F}\bar{M} = 1$) and

$$0 = \rho \frac{1-\bar{\alpha}}{1-\rho} \Gamma_c \hat{X}_t + \left(-\rho \frac{1-\bar{\alpha}}{1-\rho} \Gamma_c + \frac{1-\bar{\alpha}}{1-\rho} \Gamma_w \right) E_t \hat{X}_{t+1} \tag{102}$$

Substituting (102) into (101) yields,

$$0 = \frac{F_X}{\bar{F}} \hat{X}_t + \frac{F_{X'}}{\bar{F}} E_t \hat{X}_{t+1t} - \Gamma_w E_t \hat{X}_{t+1} \tag{103}$$

$$0 = f_X \hat{X}_t + f_{X'} E_t \hat{X}_{t+1} - \Gamma_w E_t \hat{X}_{t+1} \tag{104}$$

which is, again, identical to the Euler equation obtained in the essentially affine approximation in equation (69) except for the volatility terms. When $\sigma = 0$, perturbation and the essentially affine approximation are identical, and as before, they are also first-order equivalent local to $\sigma = 0$.

B Calculating Euler equation errors

As an example, consider the model with stochastic volatility. The calculation for the other models is analogous. Each approximation method generates a consumption function taking the form $C_t = \tilde{C}(K_{t-1}, A_{t-1}, s_{t-1}, \varepsilon_{a,t}, \varepsilon_{s,t})$. Given a value of K_{t-1}, we can calculate $\tilde{K}_t$, which is the value that arises under the consumption function $\tilde{C}$,

$$\tilde{K}_t = A_{t-1} \exp(\varepsilon_{a,t}) K_{t-1}^{\alpha} + (1-\delta) K_{t-1} - \tilde{C}(K_{t-1}, A_{t-1}, s_{t-1}, \varepsilon_{a,t}, \varepsilon_{s,t})$$

Similarly, the approximation methods imply a rule for the return on the wealth portfolio, $R_{w,t} = \tilde{R}(K_{t-1}, A_{t-1}, s_{t-1}, \varepsilon_{a,t}, \varepsilon_{s,t})$. The Euler equation error is then

$$Error\left(K_{t-1}, A_{t-1}, s_{t-1}, \varepsilon_{a,t}, \varepsilon_{s,t}\right) = E_t \left[\begin{array}{c} \beta^{\frac{1-\alpha}{1-\rho}} \left(\frac{\tilde{C}\left(\tilde{K}_t, A_t, s_t, \varepsilon_{a,t+1}, \varepsilon_{s,t+1}\right)}{\tilde{C}(K_{t-1}, A_{t-1}, s_{t-1}, \varepsilon_{a,t}, \varepsilon_{s,t})} \right)^{-\rho\frac{1-\alpha}{1-\rho}} \times \\ R\left(\tilde{K}_t, A_t, s_t, \varepsilon_{a,t+1}, \varepsilon_{s,t+1}\right)^{\frac{\rho-\alpha}{1-\rho}} \times \\ \left(\alpha A_t \exp\left(\varepsilon_{a,t+1}\right) \tilde{K}_t^{\alpha-1} + 1 - \delta\right) \end{array} \right] - 1$$

where $A_t = A_{t-1} \exp(\varepsilon_{a,t})$ and $s_t = (1 - \phi_s) + \phi_s s_{t-1} + \varepsilon_{s,t}$. The error is a function of the state variables as of date $t-1$ and the realizations of the shocks on date t. I simulate the state variables under the three approximations and then calculate the errors along the simulated paths. To calculate the expectations, I use Gaussian quadrature over the shocks with 15 abcissas. The figures plot histograms of $\log_{10} Error\left(K_{t-1}, A_{t-1}, s_{t-1}, \varepsilon_{a,t}, \varepsilon_{s,t}\right)$.

Table 1. Calibration

Common parameters			
γ	0.33	δ	0.02
ϱ	0.66	β	0.99
Stochastic volatility			
α	15	σ_s	0.05
φ_s	0.95	σ_a	0.01
Time-varying risk aversion			
$\bar{\alpha}$	15	σ_α	1.66
φ_α	0.95	σ_a	0.01
Time-varying disaster risk			
p	0.005	σ_p	0.000825
φ_p	0.92	B	0.4

Note: the top section gives parameters that are common to all three experiments. The remaining three sections give the parameters specific to the individual calibrations

Figure 1. log$_{10}$ Euler equation errors with stochastic volatility

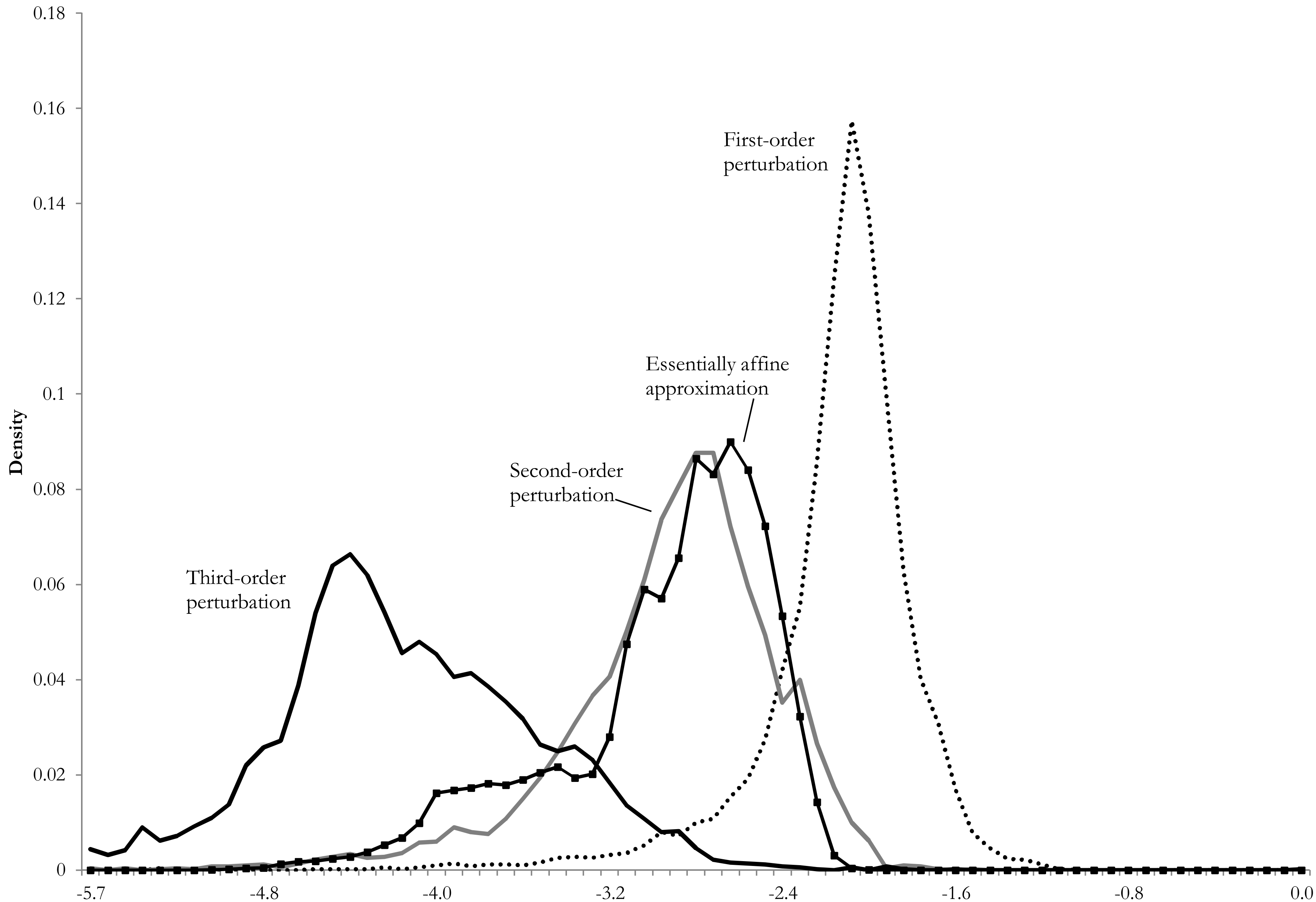


Figure 2. $\log_{10}$ Euler equation errors with time-varying risk aversion

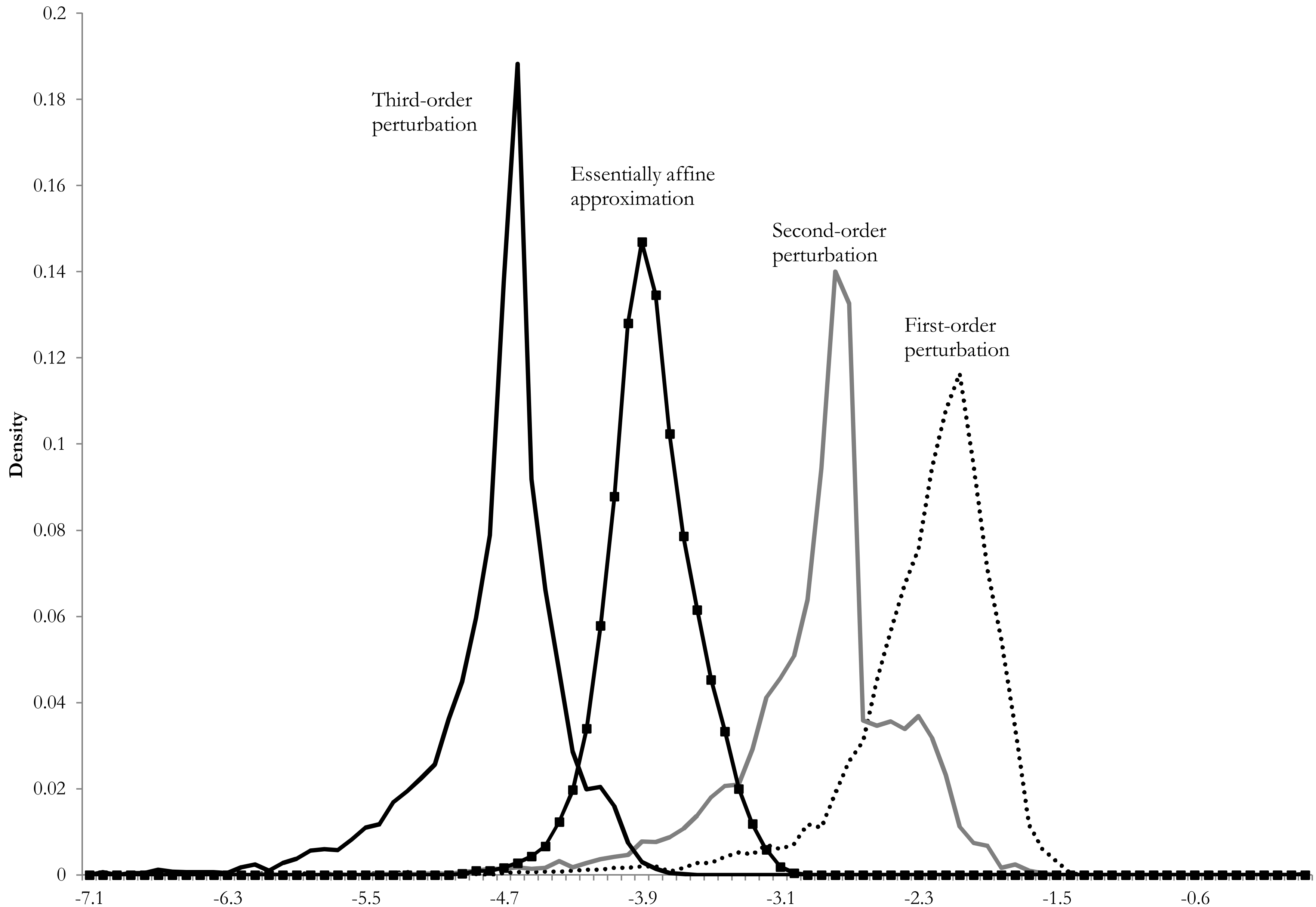
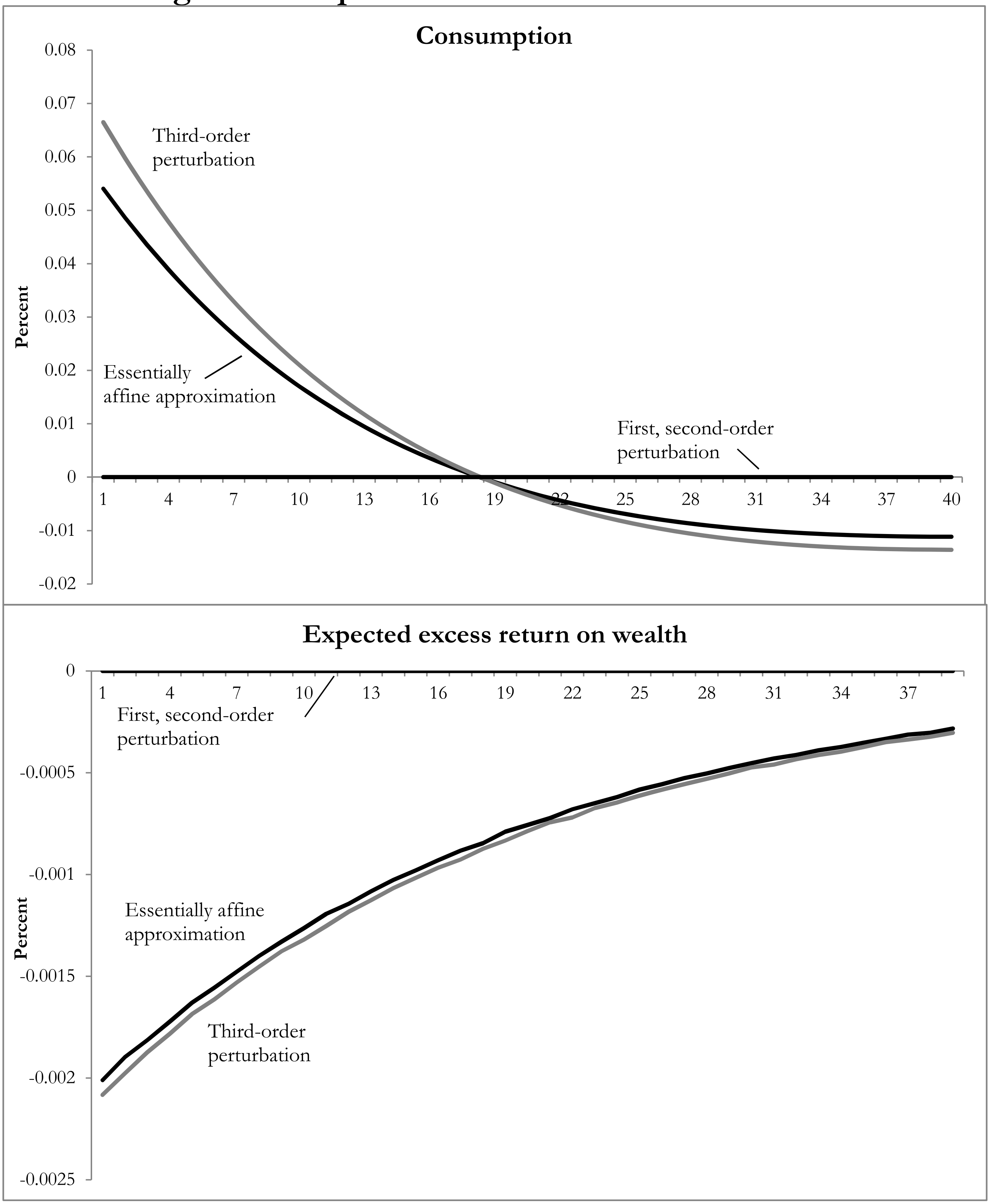


Figure 3. Responses to a decline in risk aversion



Note: responses to a unit standard-deviation decrease in risk aversion under the four approximations. The top panel plots consumption, the bottom panel the expected excess return on the household's wealth (a claim to aggregate consumption) above the risk-free rate. IRF are calculated from the stochastic steady states (where the model returns if all shocks equal zero).